Continuous One-Counter Automata

Philip Offtermatt

Joint work with:
Michael Blondin, Tim Leys,
Filip Mazowiecki, Guillermo Pérez

MAX PLANCK INSTITUTE
FOR SOFTWARE SYSTEMS

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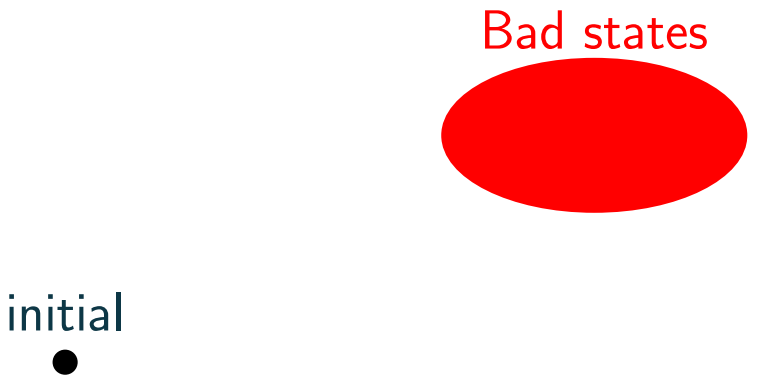
Why we care about overapproximations

Goal: Verify no **bad state** is **reachable**!



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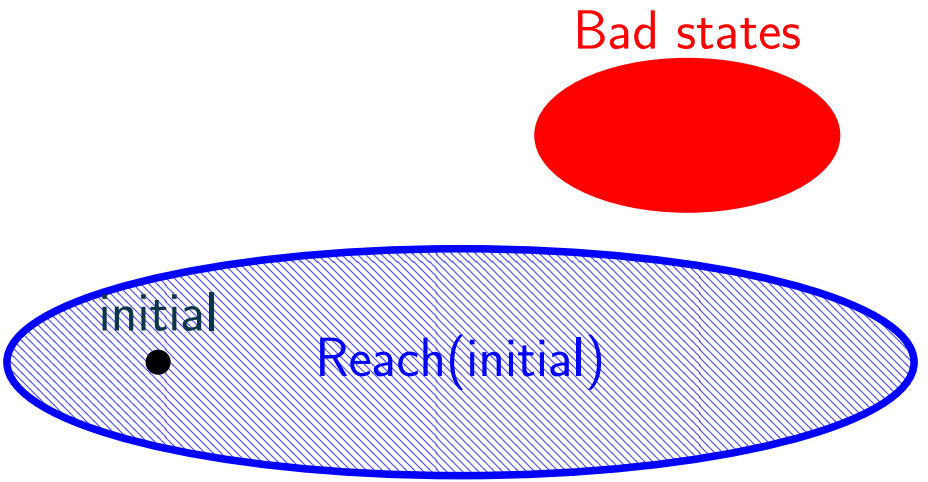

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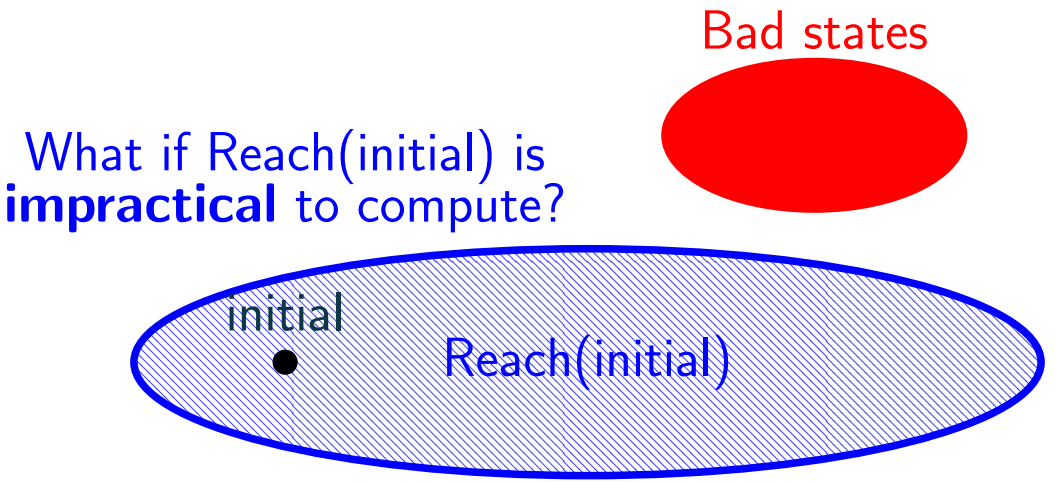

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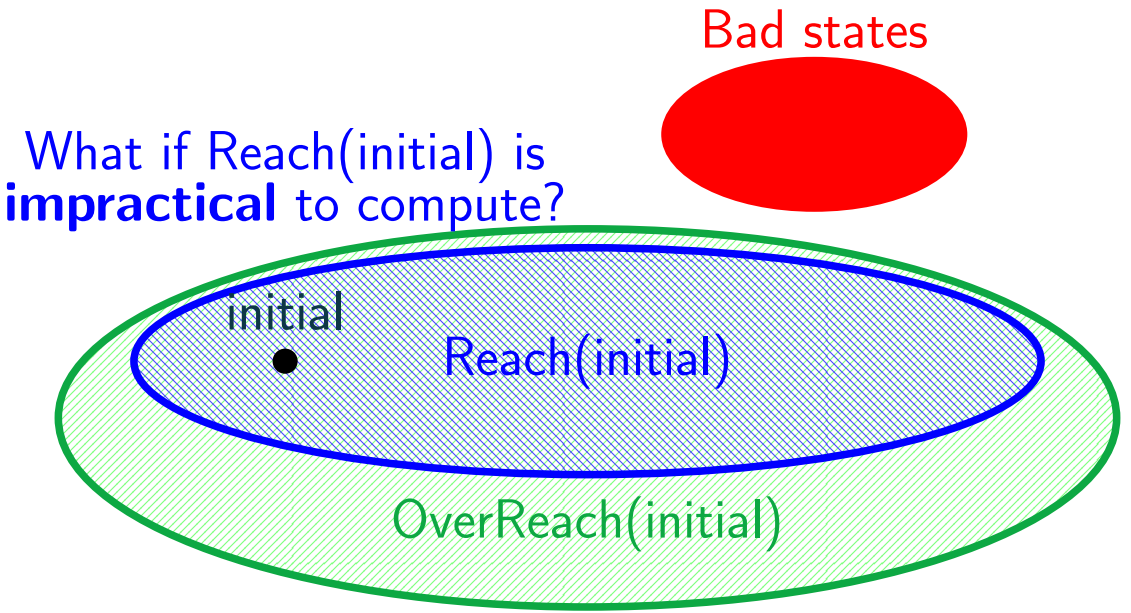

Compute an **Overapproximation** instead!

Overapproximations help proving safety

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Goal: Find efficient overapproximations for models with hard reachability problems

Overapproximations help proving safety

Goal: Find efficient overapproximations
for models with hard reachability problems

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models representing interesting systems

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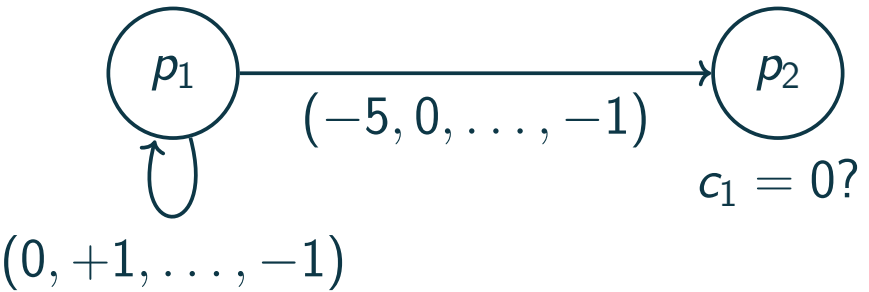
One-Counter Automata

Counter Automata



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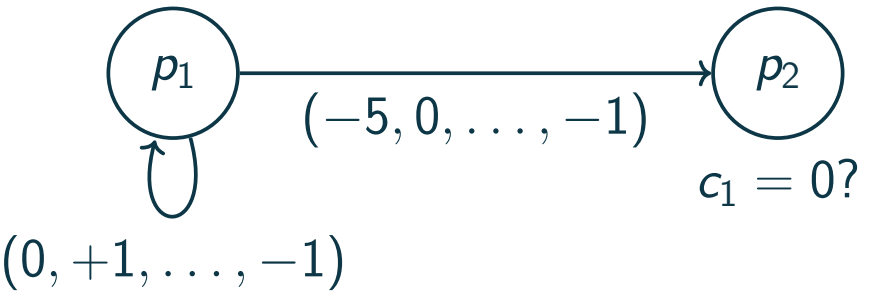
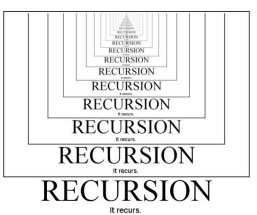


Represent complex systems



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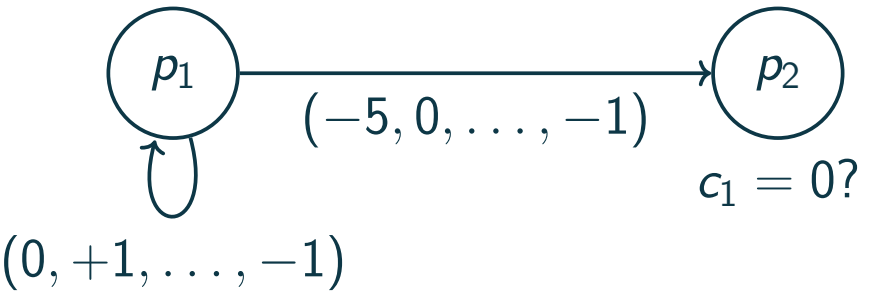

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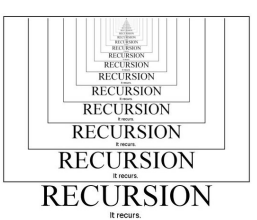

But **Reachability is undecidable!**

...even with only two counters [Minsky,'61]

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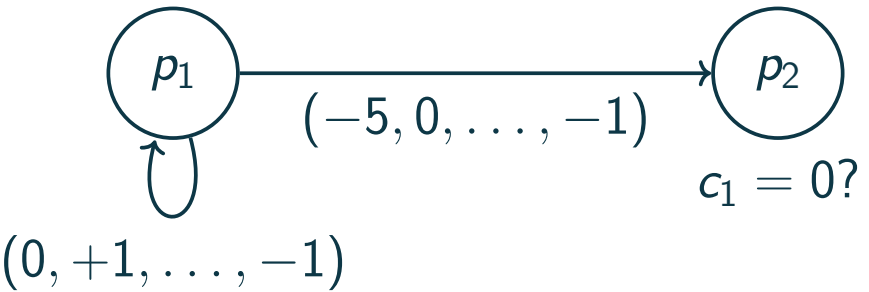

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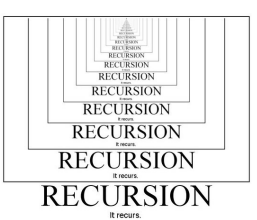

But **Reachability is undecidable!**
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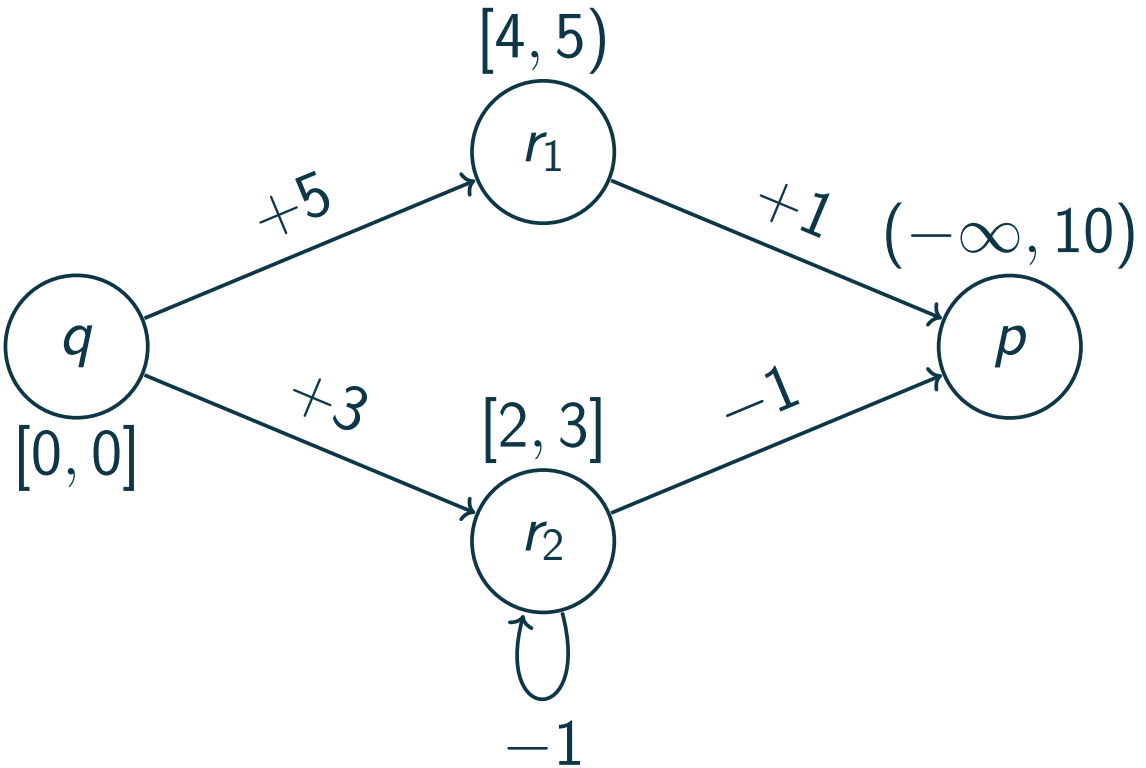
$\Rightarrow$ Restrict to **One**-Counter Automata!

One-Counter Automata (OCA)



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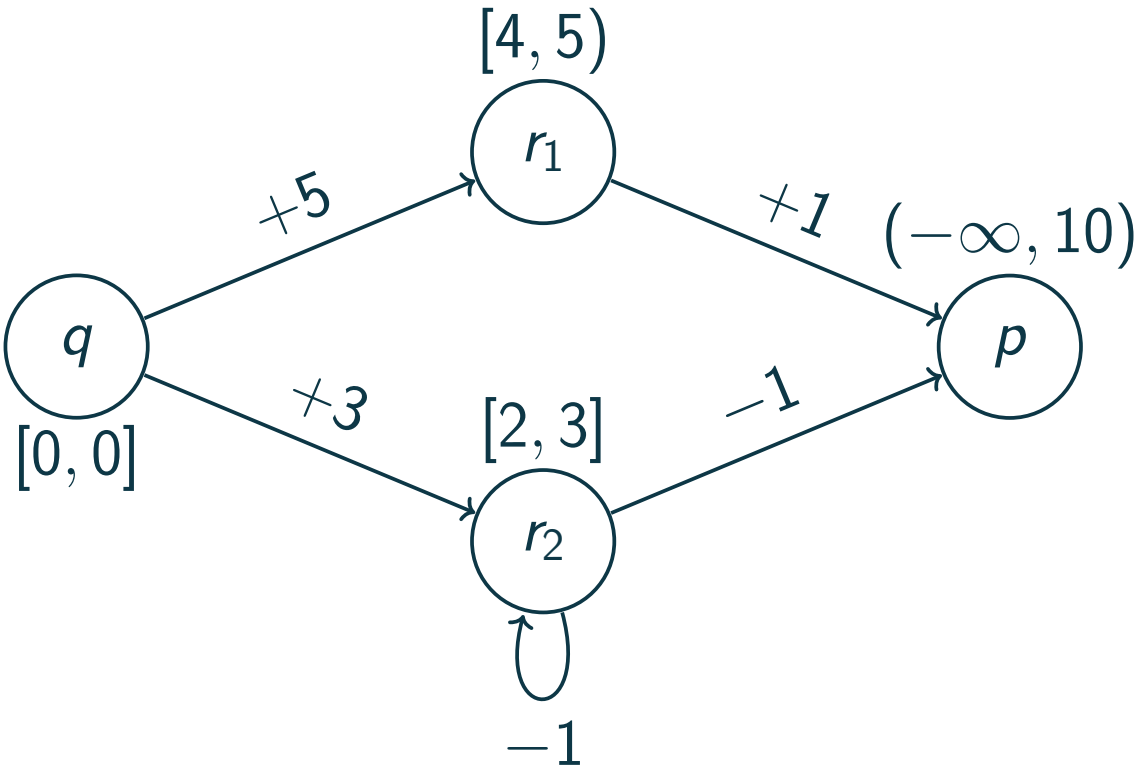

Run: $q(0)$

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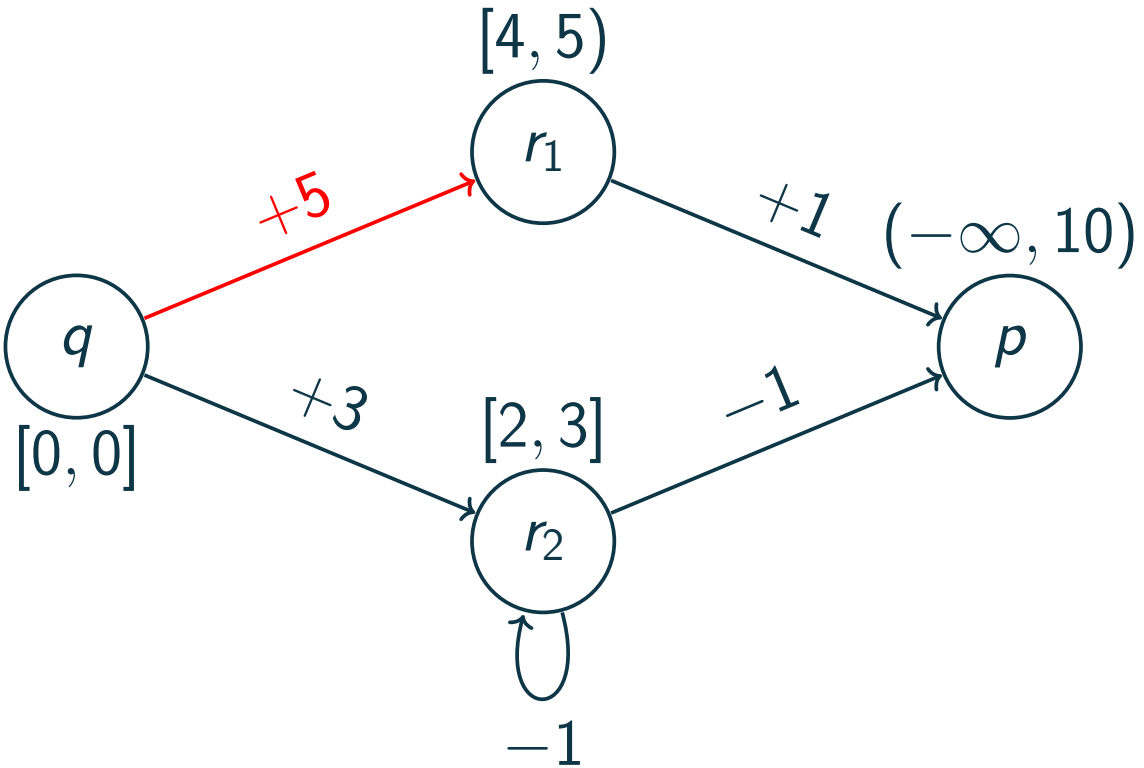

Run: $q(0) \rightarrow r_1(5)$

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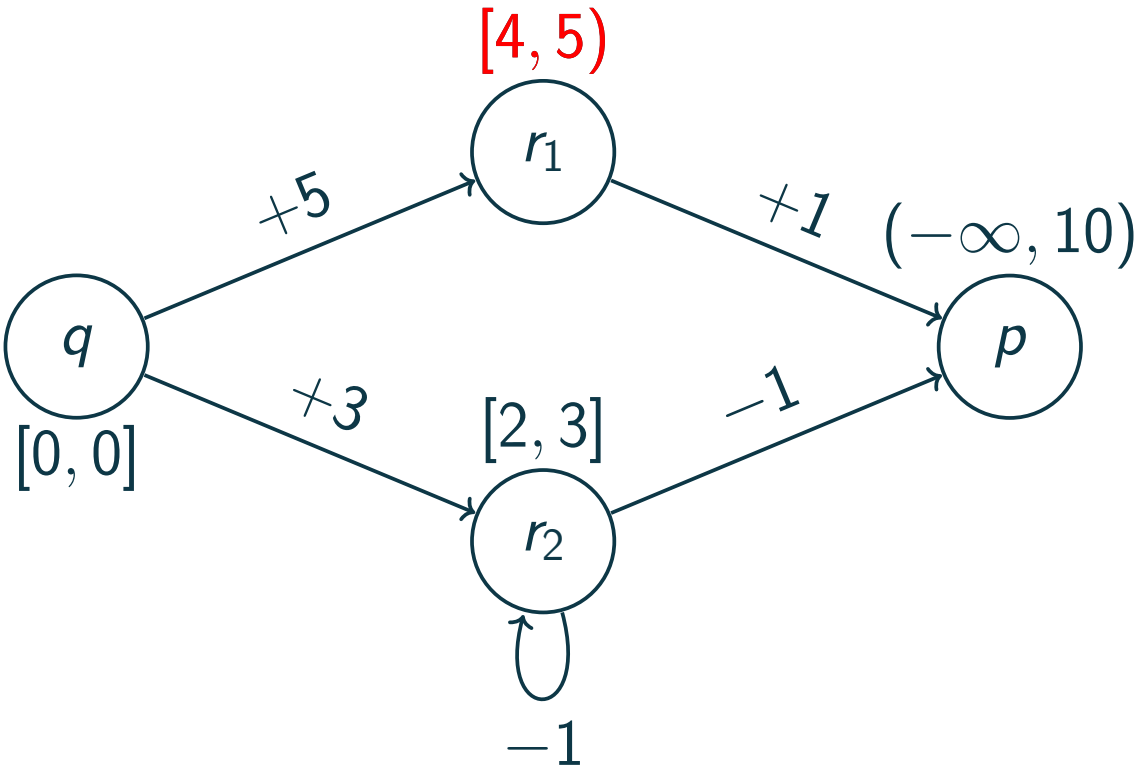

Run: $q(0) \rightarrow \cancel{r_1(5)}$

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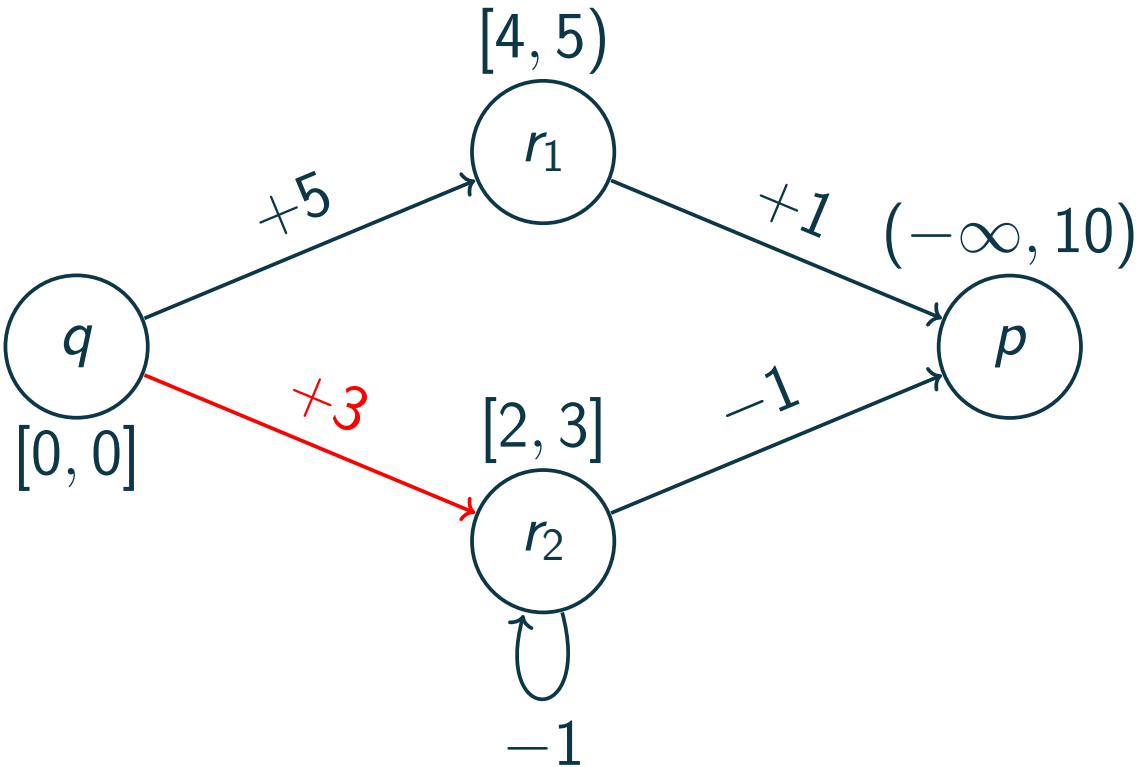

Run: $q(0) \to r_1(5) \to r_2(3)$

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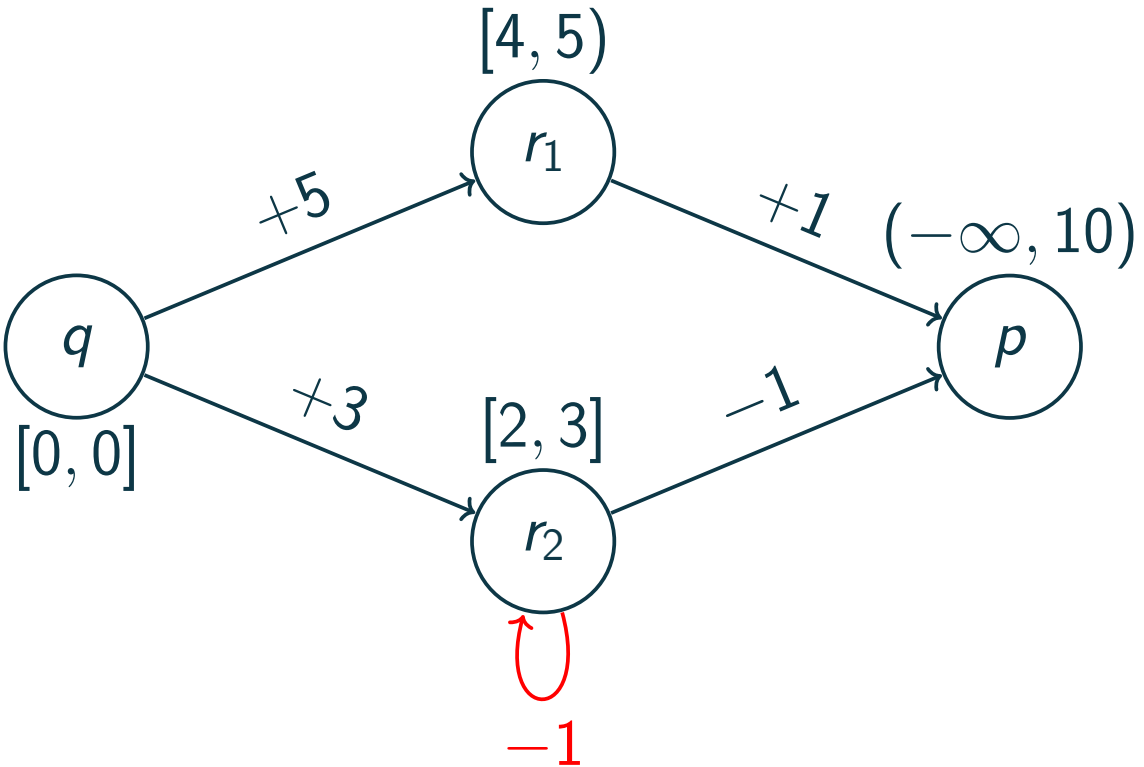

Run: $q(0) \not\to r_1(5) \to r_2(3) \to r_2(2)$

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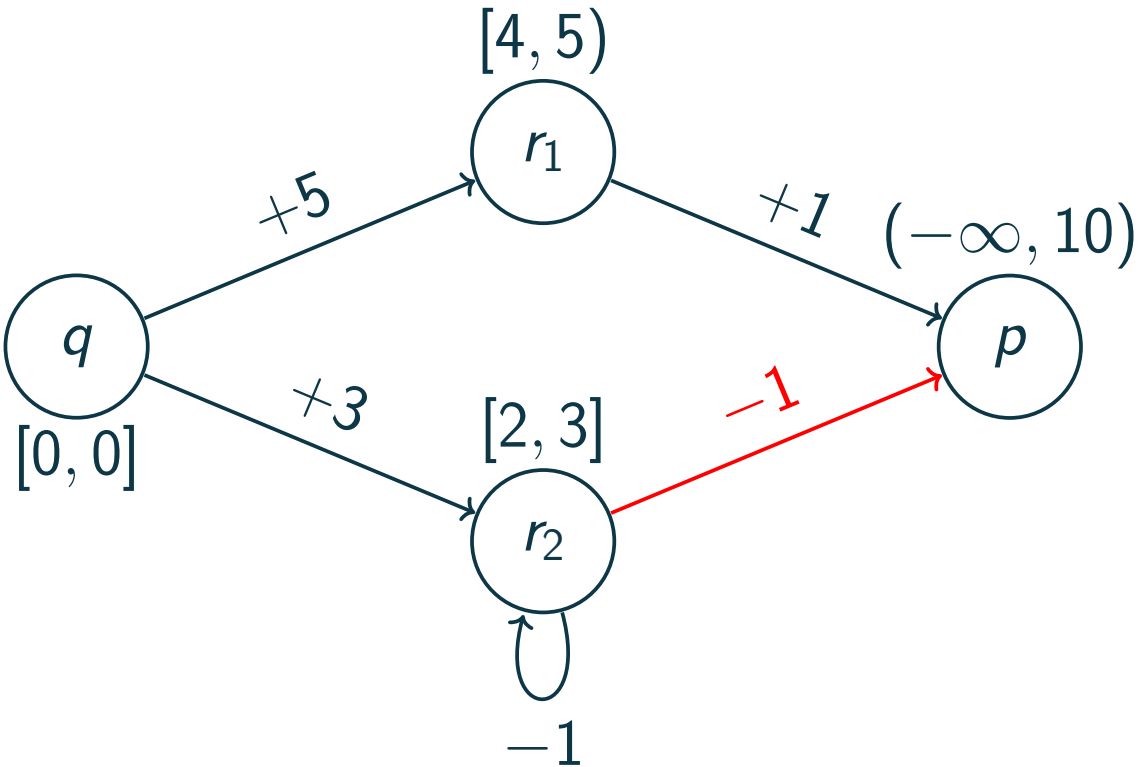
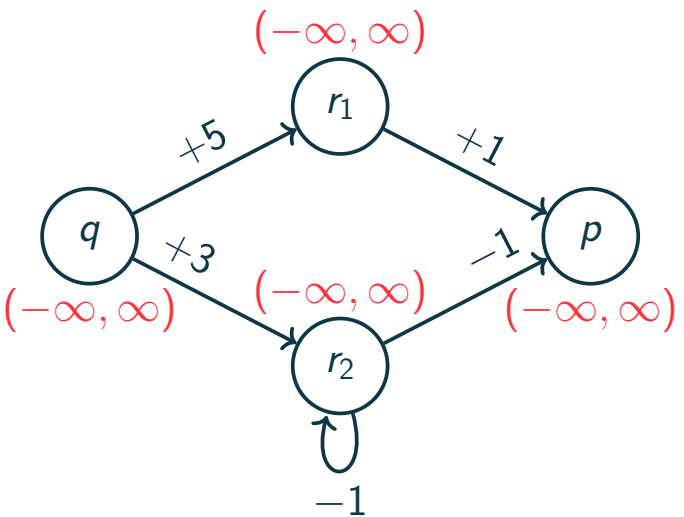


$\rightarrow r_2(3) \rightarrow r_2(2) \rightarrow p(1)$

Run: $q(0) \not\rightarrow r_1(5)$

Variants of One-Counter Automata (OCA)

Guardless OCA



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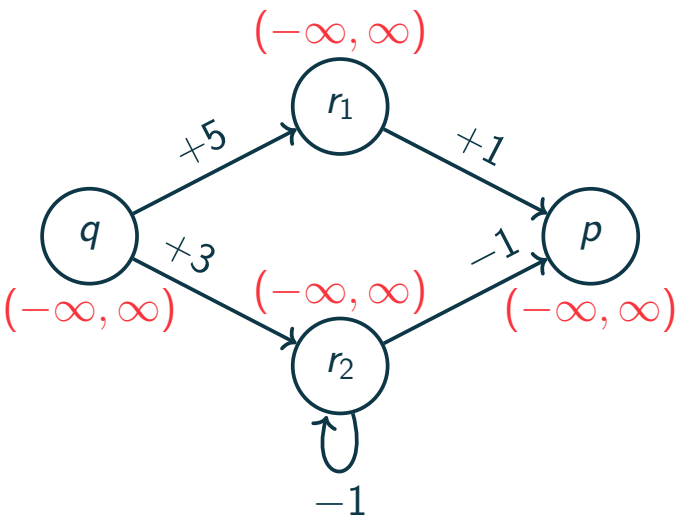
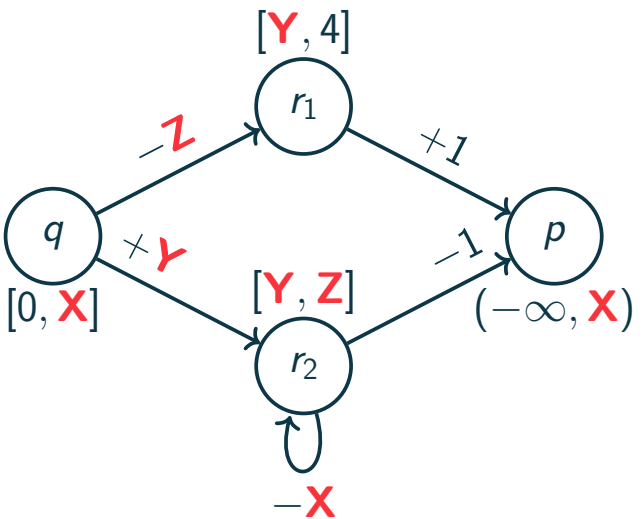


Parametric OCA



Reachability in One-Counter Automata (OCA)

What is the state-of-the-art?

Guardless OCA:
NP-complete
[Haase et al.,'09]

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Our goal:
Overapproximate OCA efficiently!

Contributions

Propose novel model:

Continuous One-Counter Automata (COCA)

Overapproximation for OCA with tractable complexity

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Prove complexity results:

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Prove complexity results:

Reachability in **guardless COCA**:
In $\mathbf{NC}^2$

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Reachability in **standard COCA**:
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Overapproximation for OCA with tractable complexity

Prove complexity results:

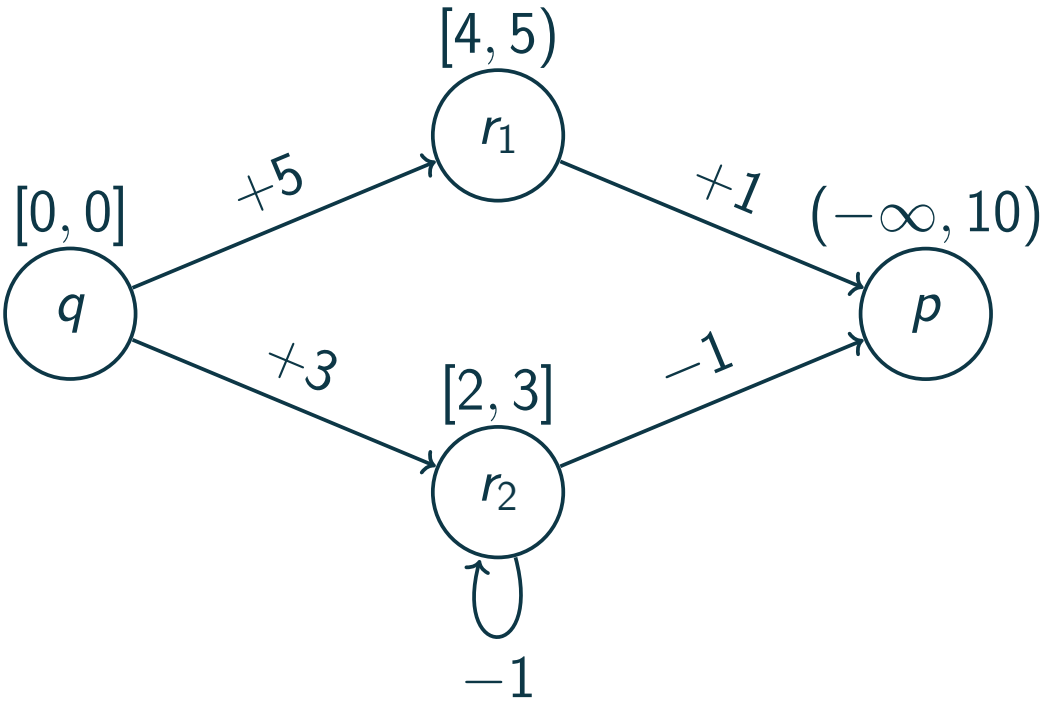
Reachability in **guardless COCA**:
In **NC^2**

Reachability in **standard COCA**:
In **P-time**

Reachability in **parametric COCA**:
NP-complete

Overapproximating One-Counter Automata

Continuous One-Counter Automata (COCA)



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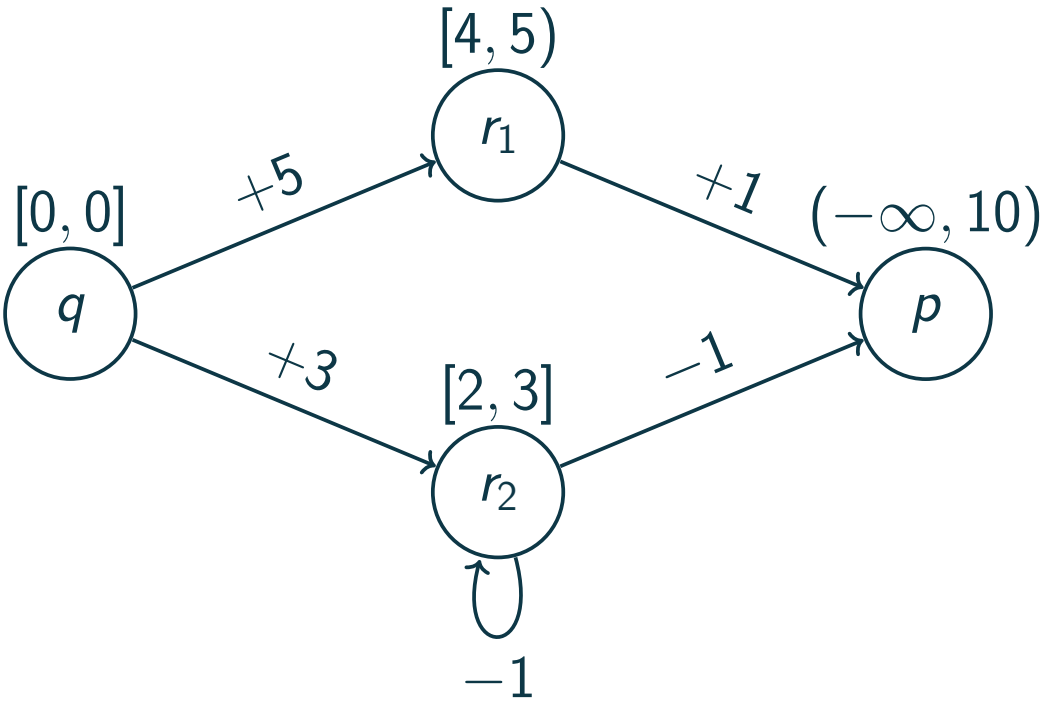

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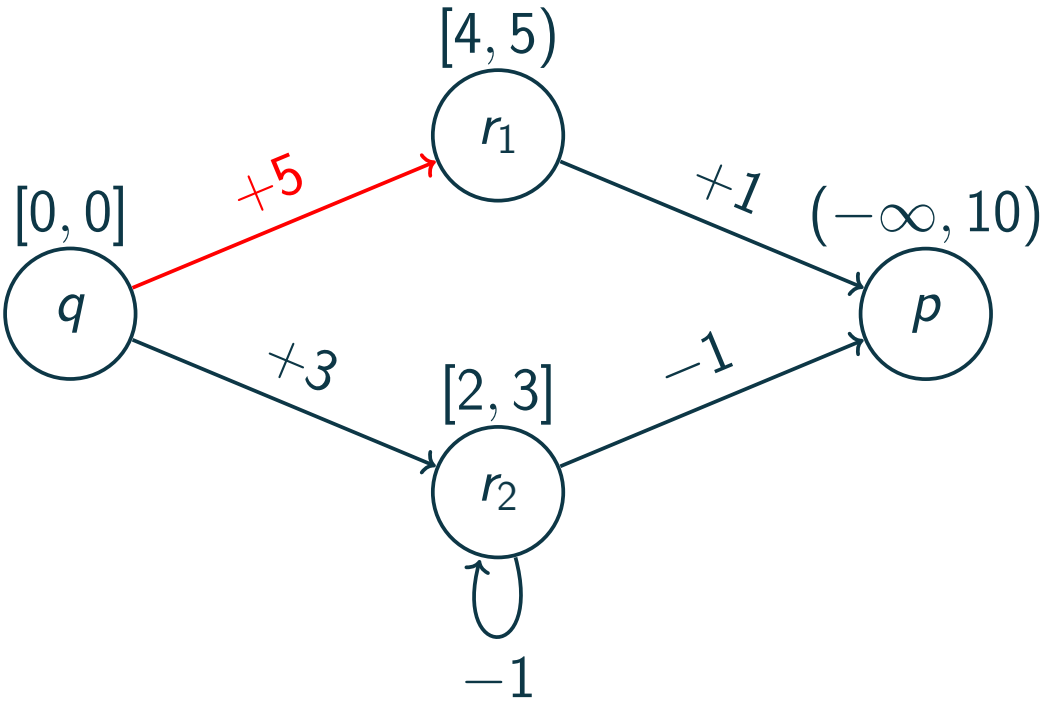

Run: $q(0) \xrightarrow{4/5} r_1(4)$

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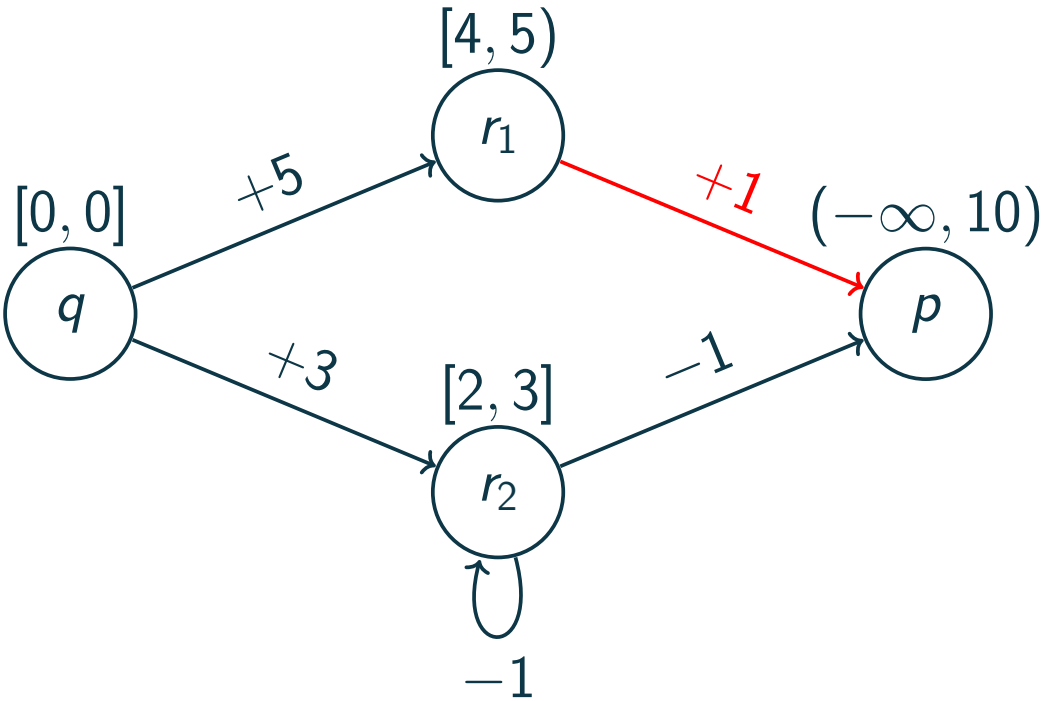

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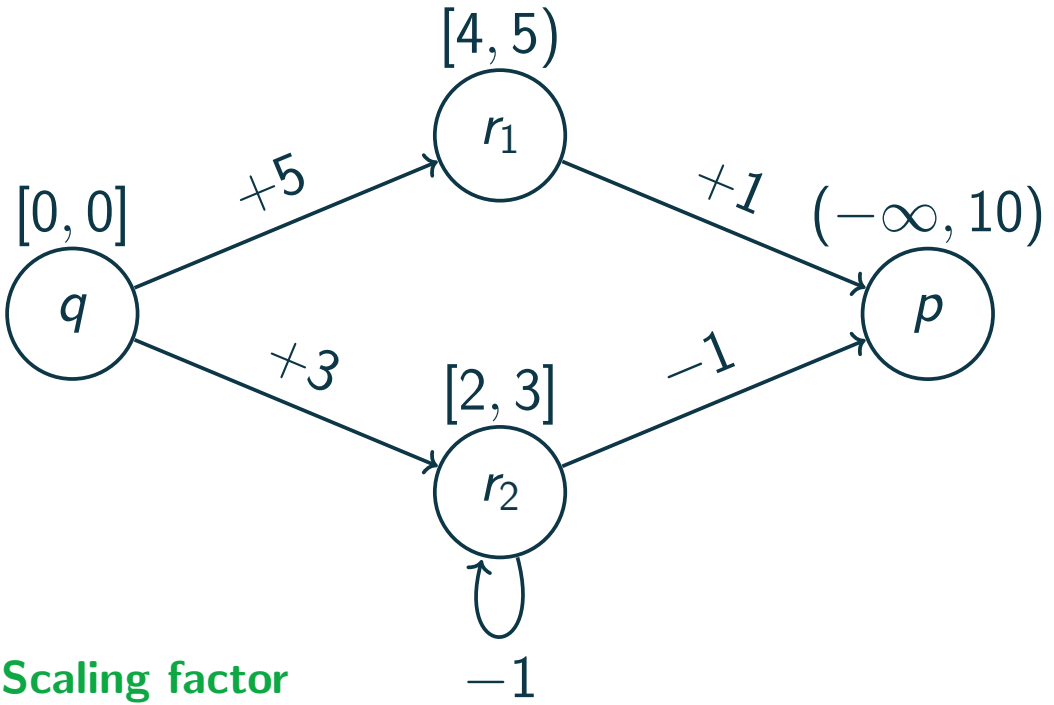

Scaling factor

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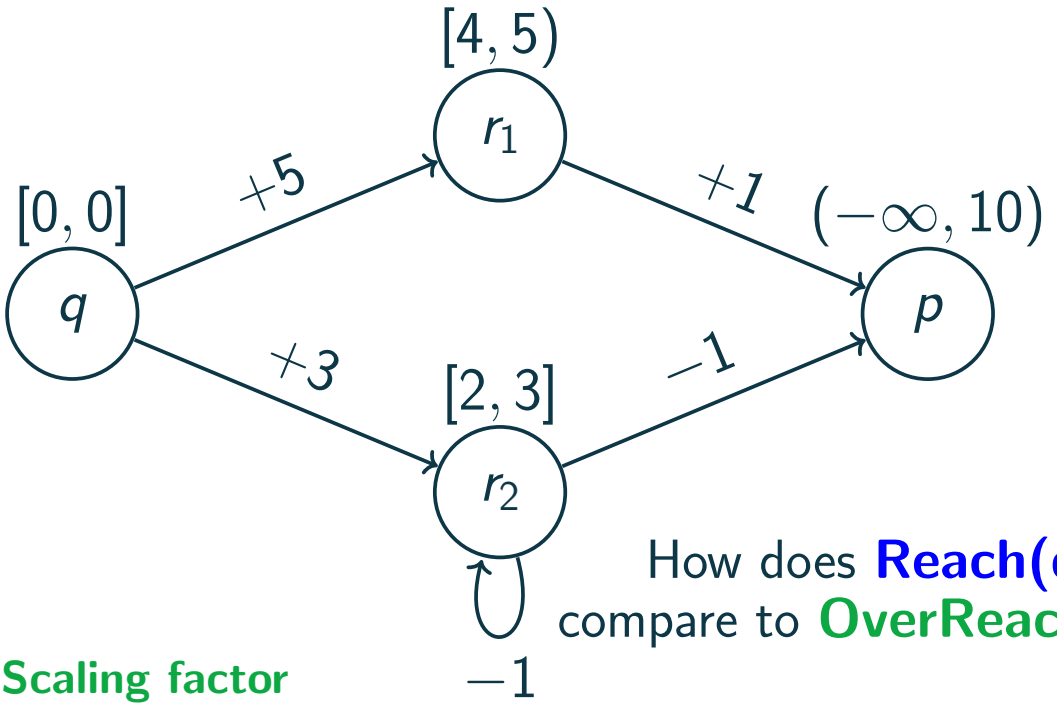

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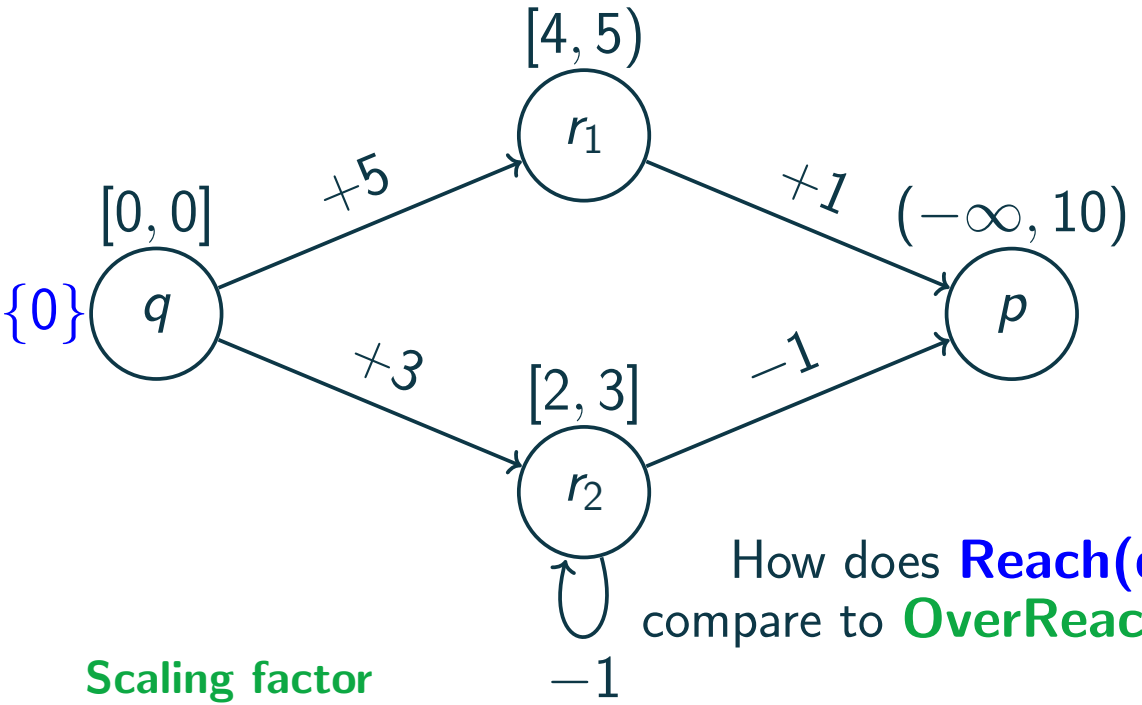

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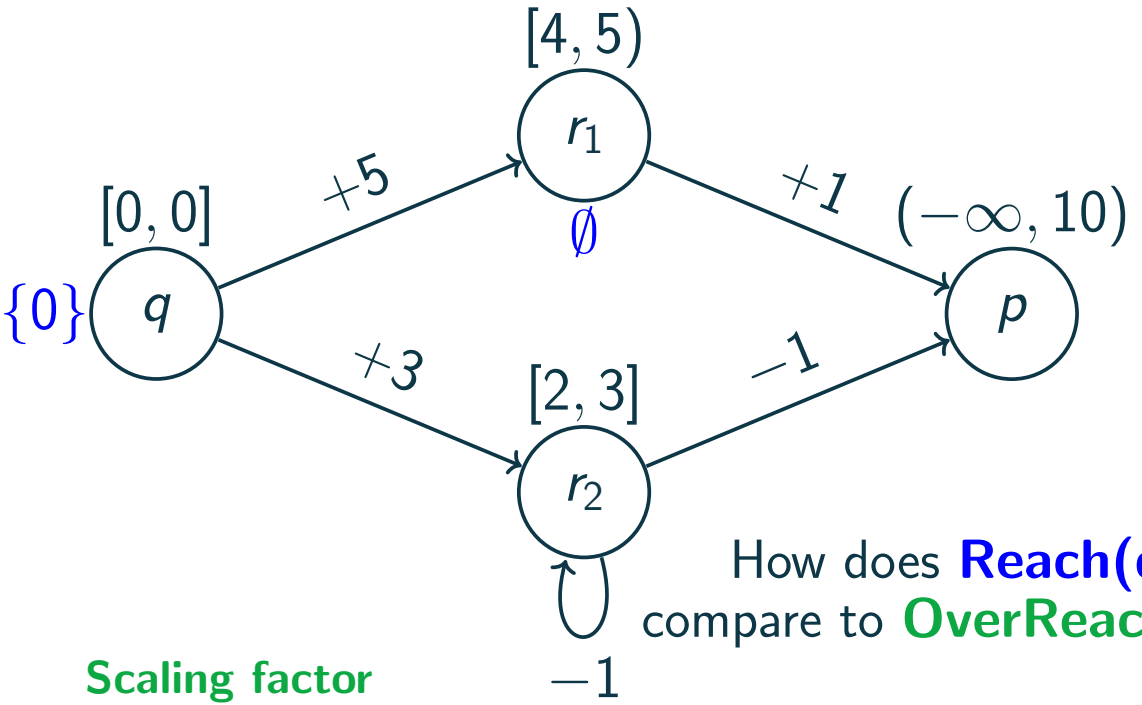

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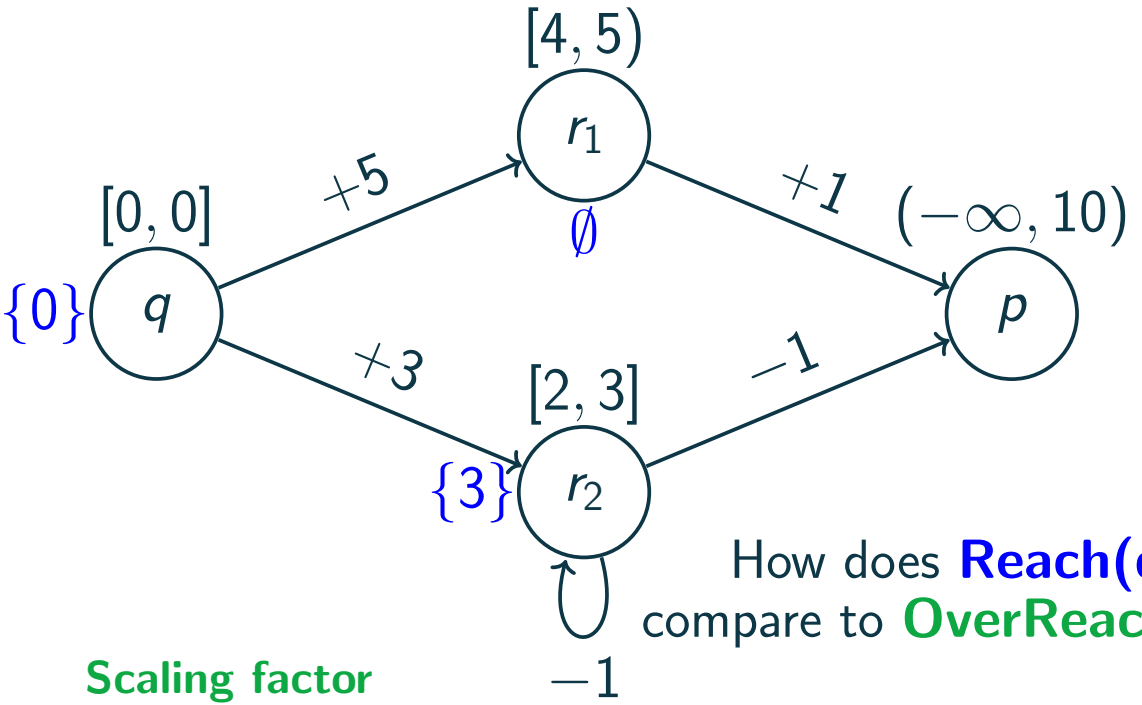

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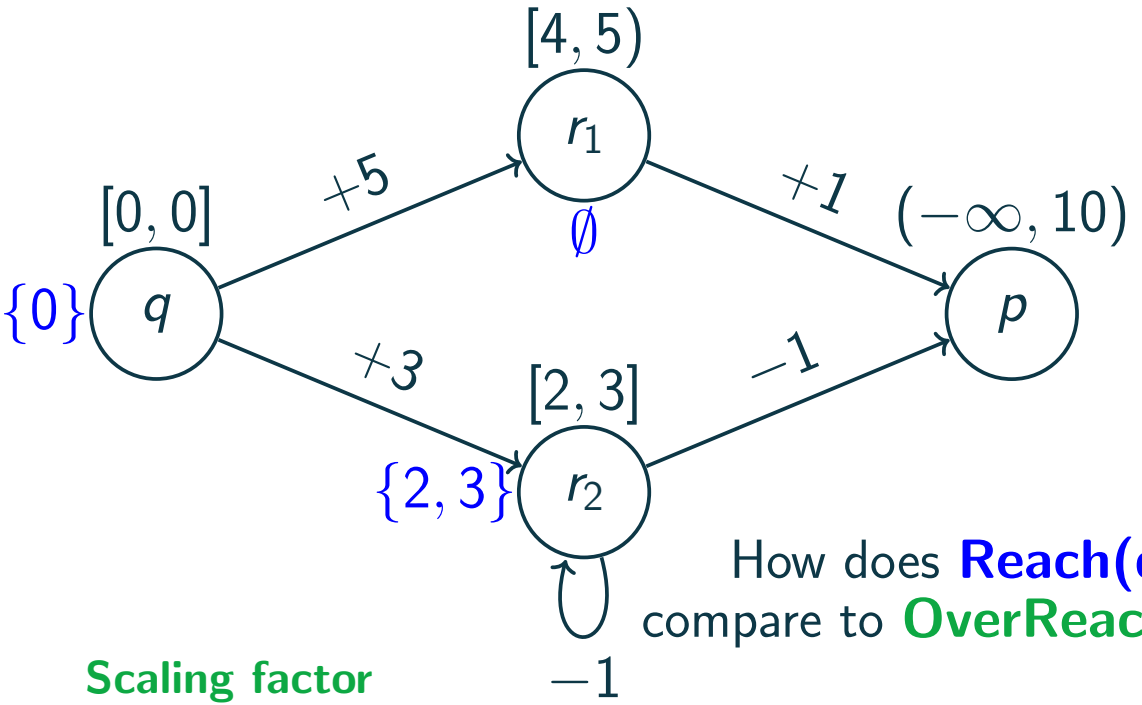

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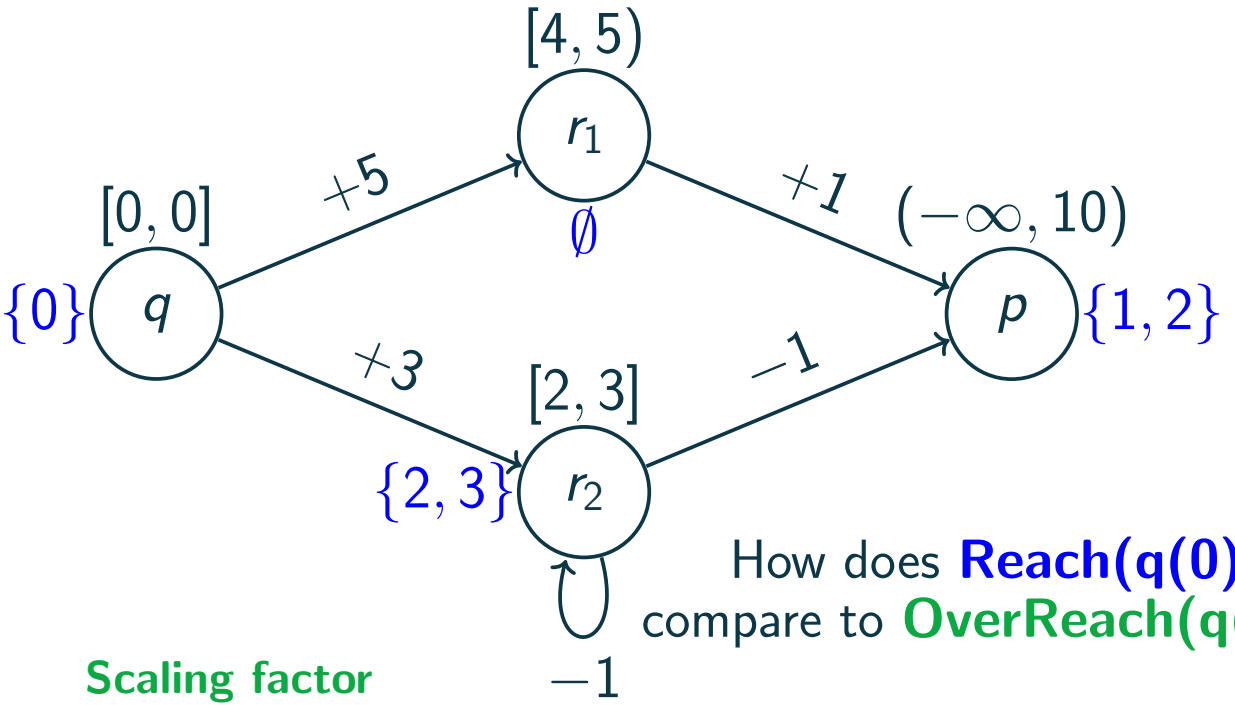

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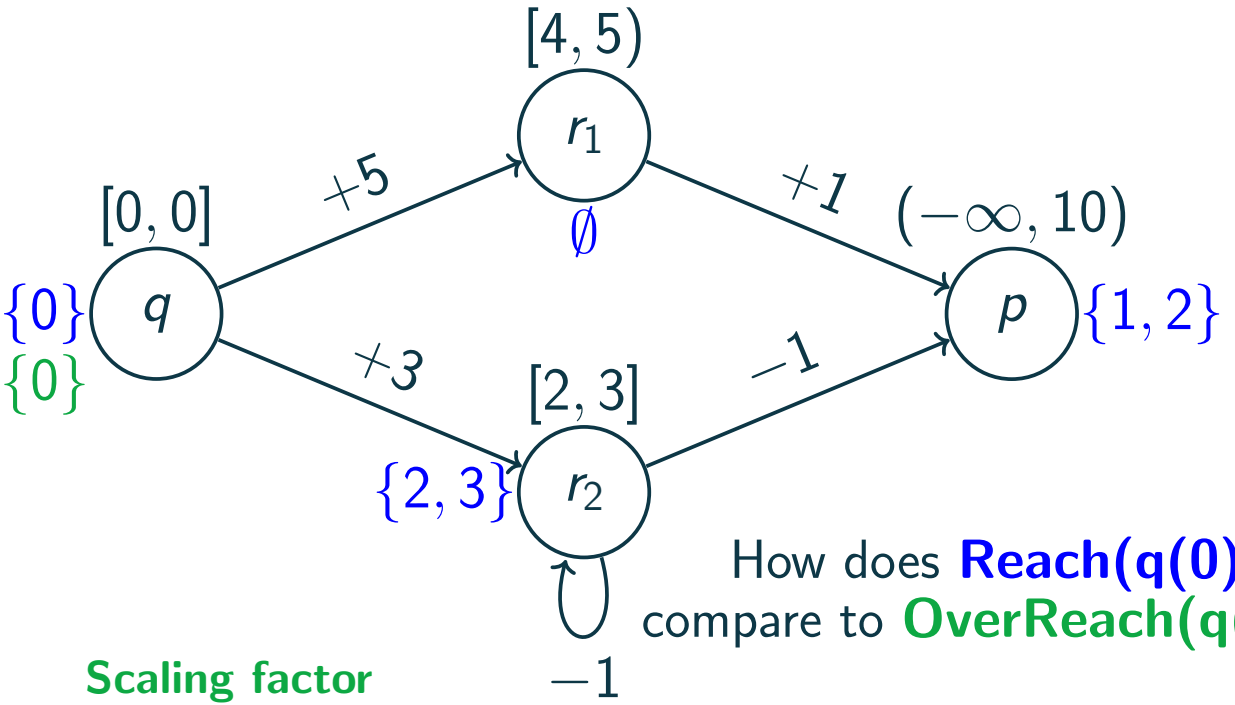

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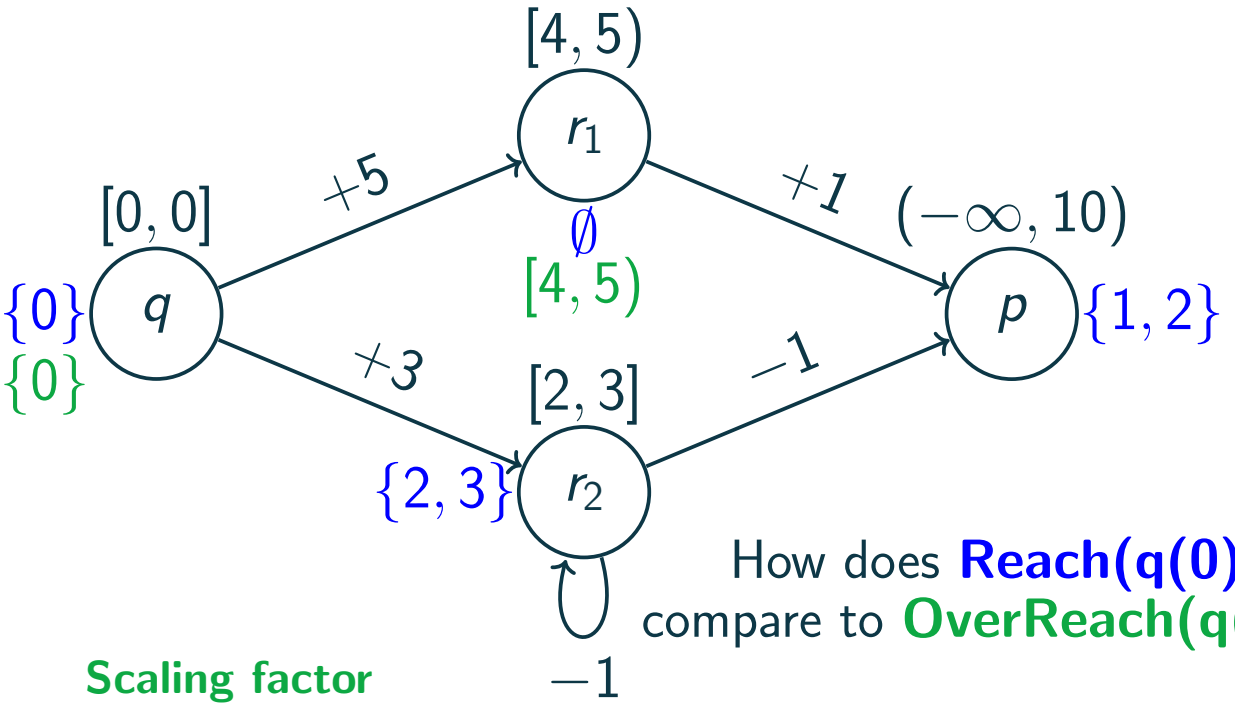

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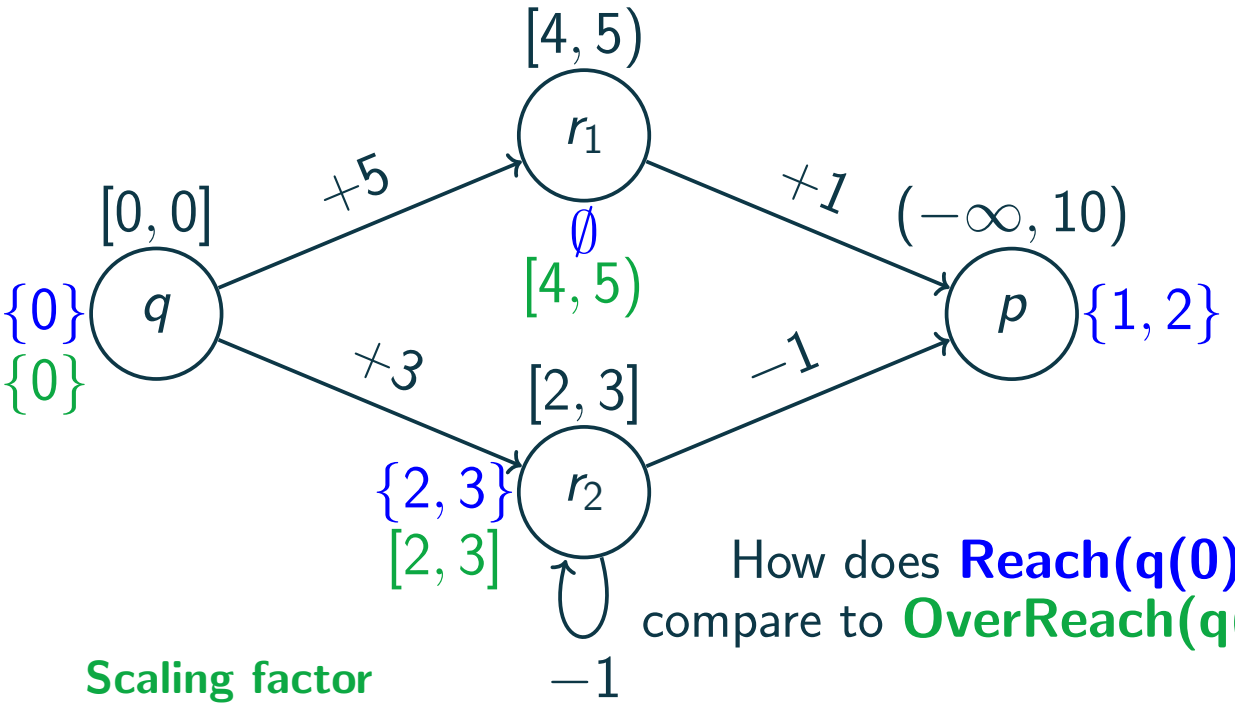

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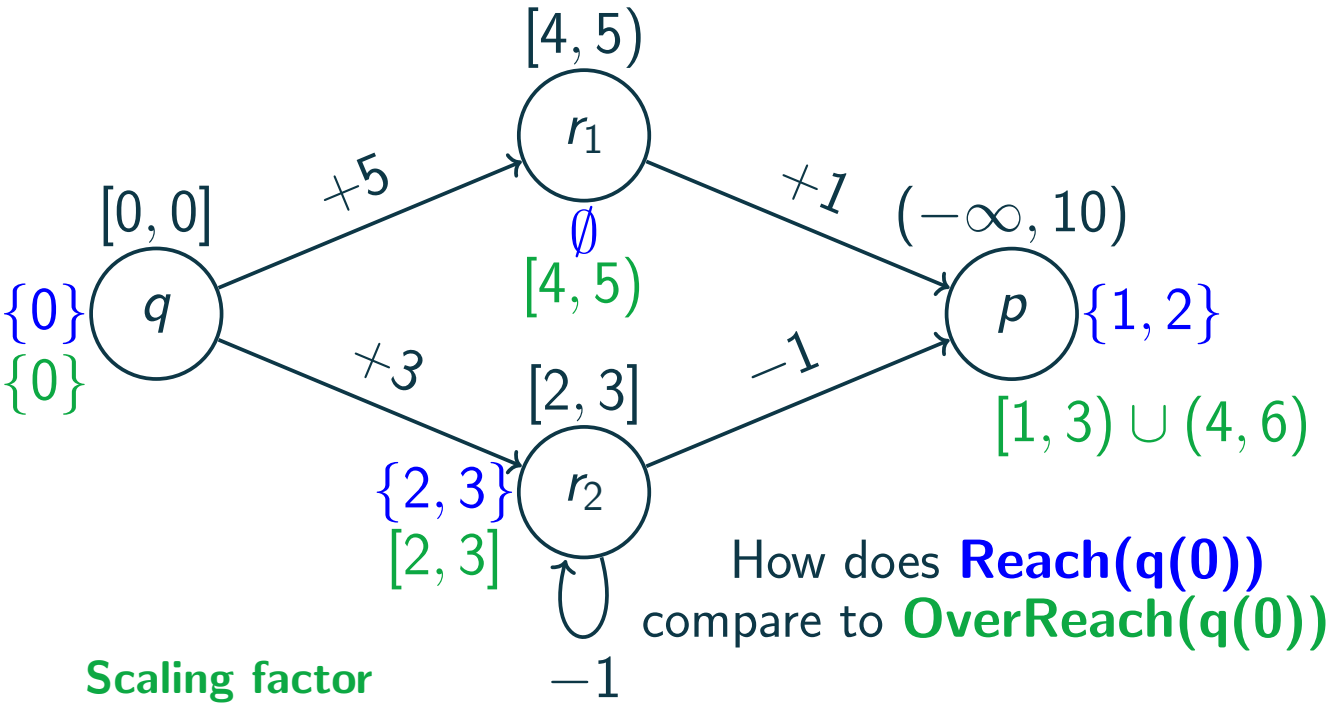

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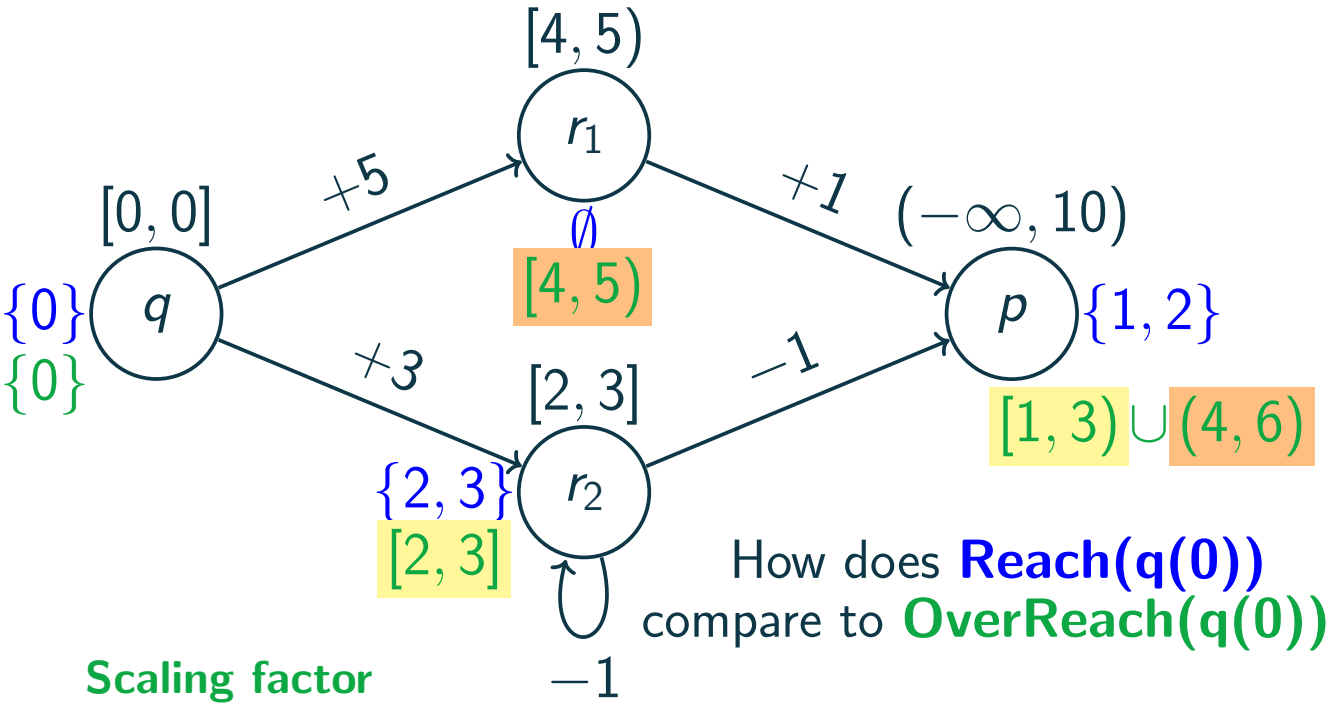

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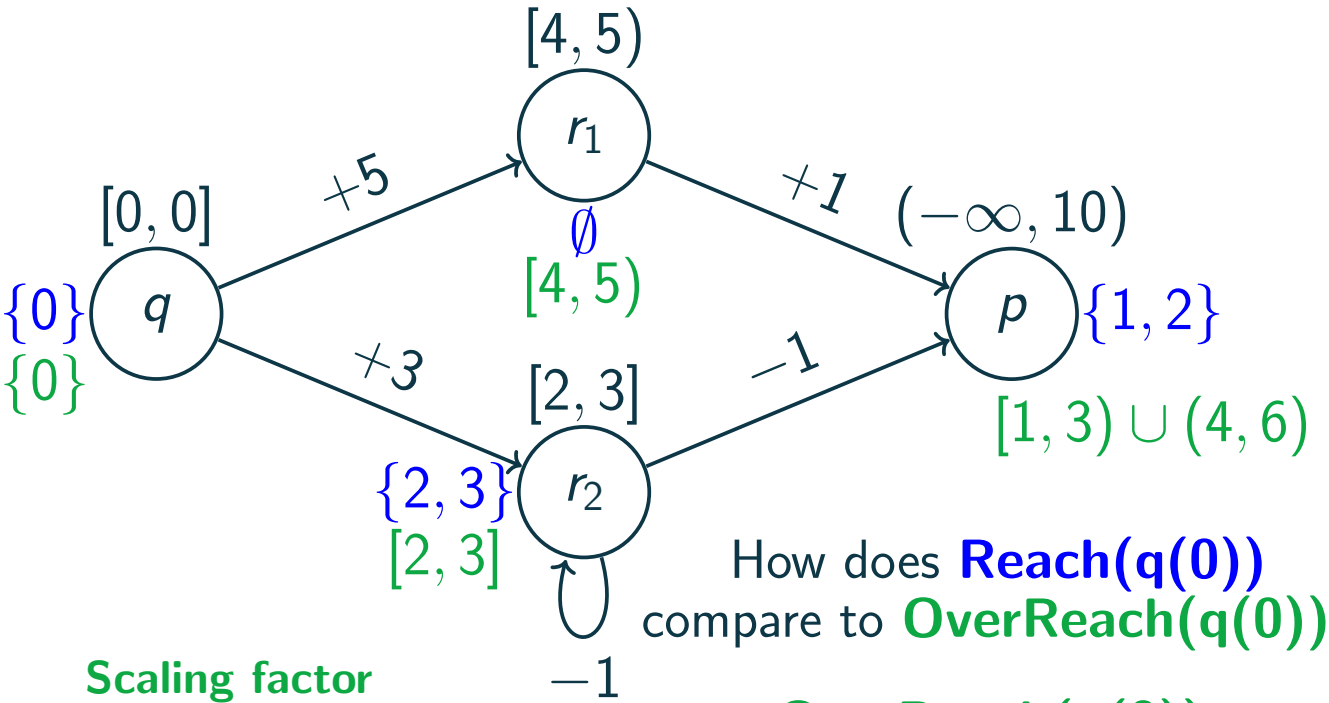

How does **Reach(q(0))** compare to **OverReach(q(0))**?

OverReach(q(0)): union of (few) intervals

Scaling factor

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Reachability in Continuous OCA (COCA)

...has much lower complexity

Guardless OCA:
NP-complete
[Haase et al.,'09]

$\Rightarrow$

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NC^2: **Polynomially** many **random-access machines** running for at most **$O(\log^2 n)$** steps in parallel

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Reachability in Continuous OCA (COCA)

...has much lower complexity

Guardless OCA:
NP-complete
[Haase et al.,'09]

$\Rightarrow$

Guardless COCA:
in $\mathbf{NC^2}$ (below P-time)

OCA:
PSPACE-complete
[Fearnley,Jurdziński,'15]

$\mathbf{NC^2}$: **Polynomially** many **random-access machines** running for at most $\mathbf{O(\log^2 n)}$ steps in parallel

Parametric OCA:
Decidability unknown

Notably:
Graph reachability $\in \mathbf{NC^2}$
Also for weighted graphs!

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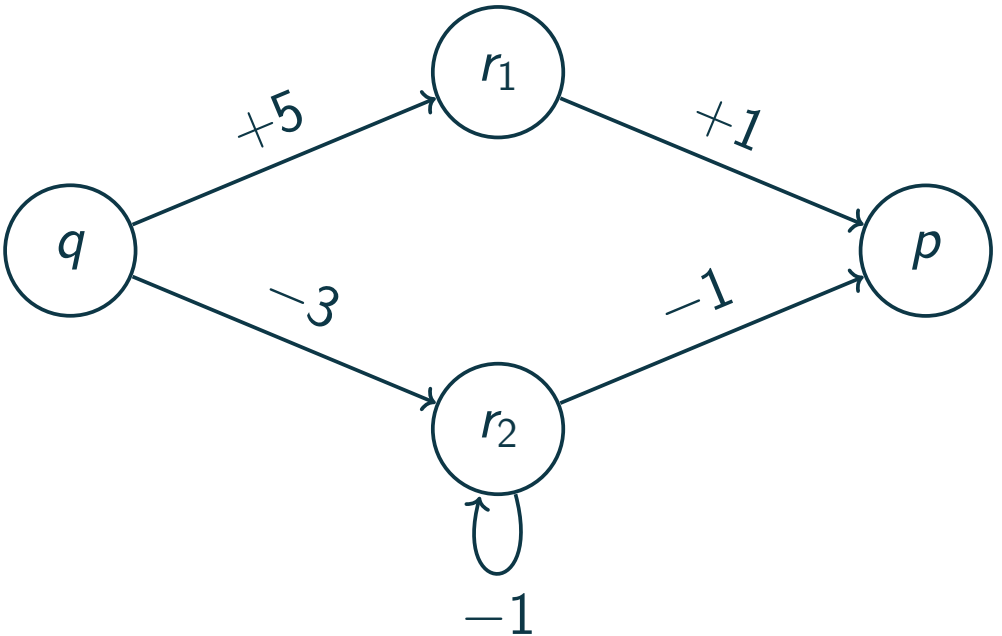

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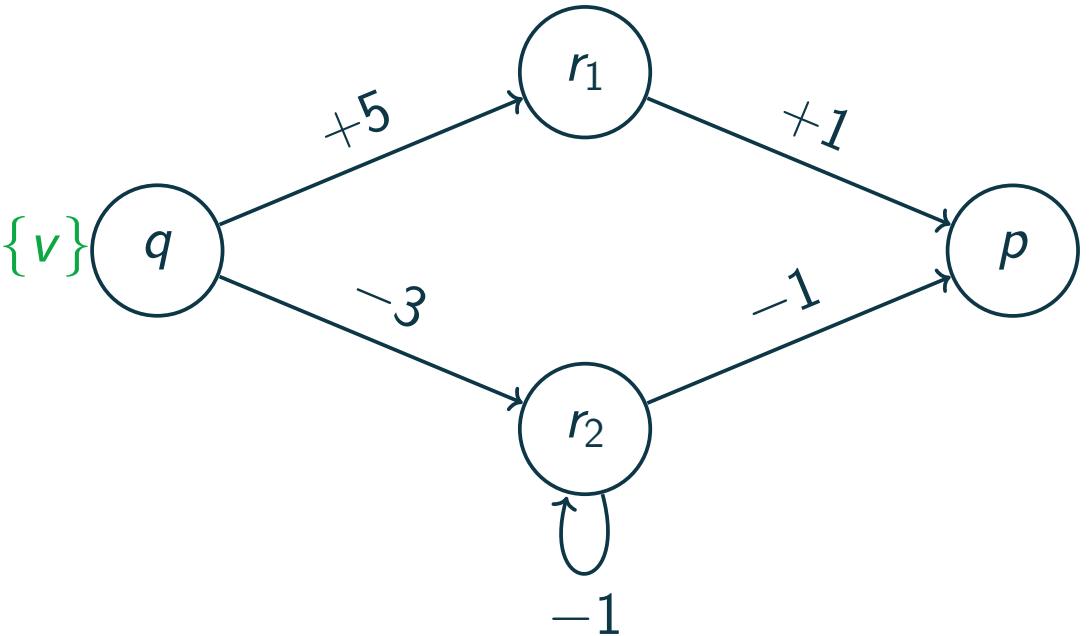

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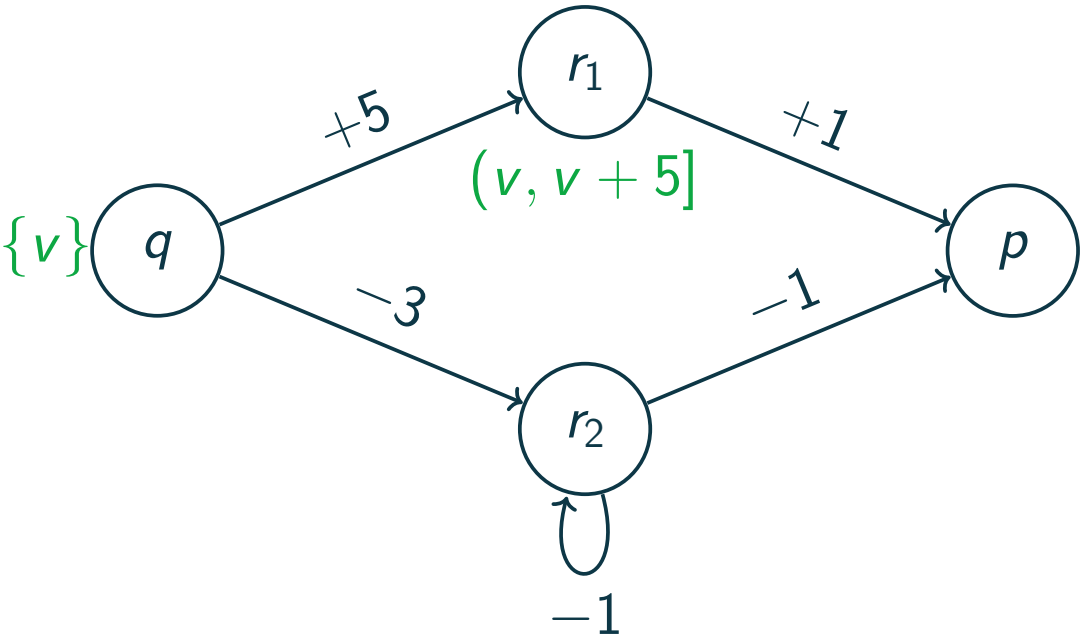

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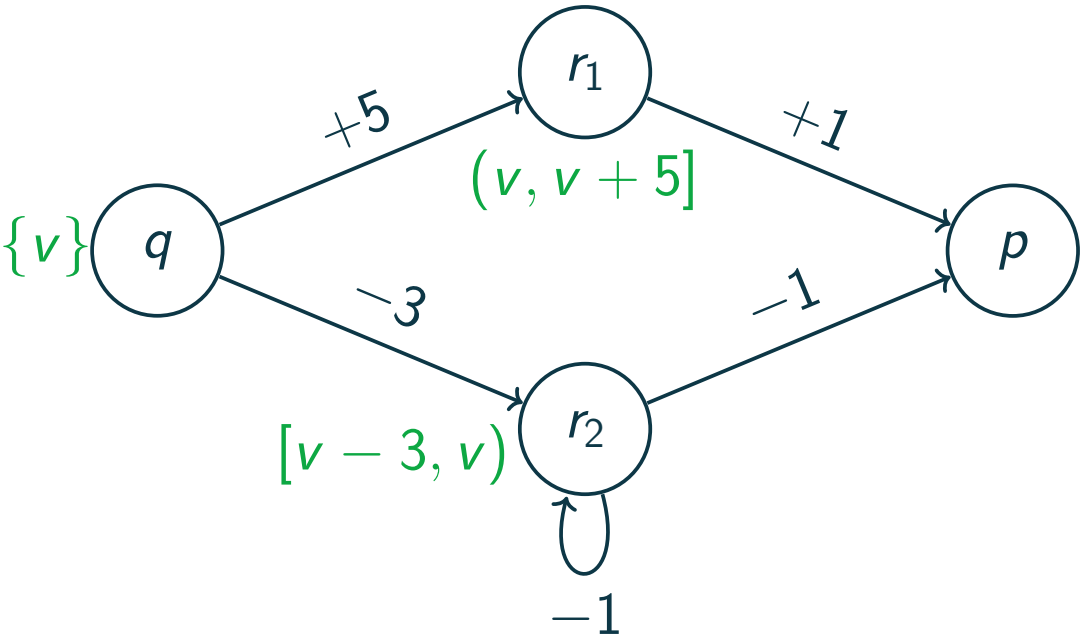

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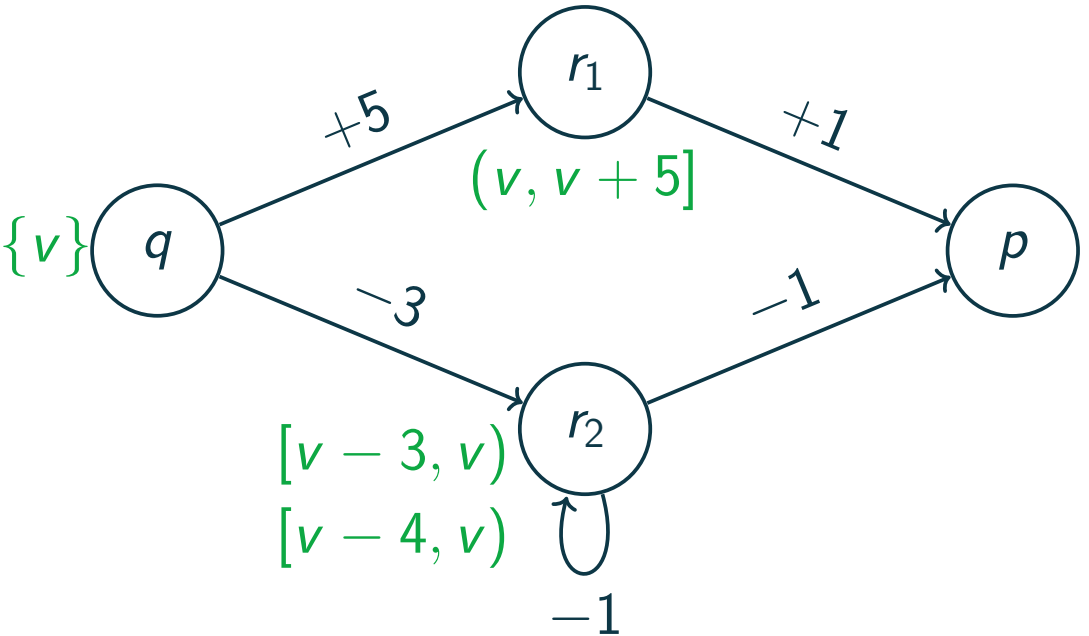

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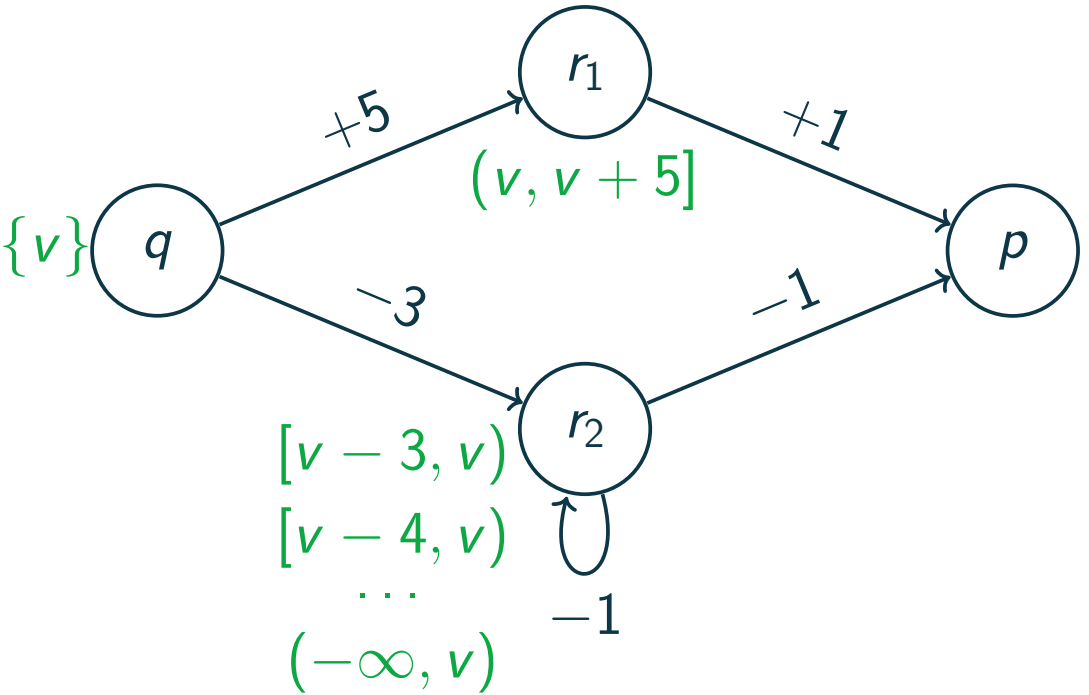

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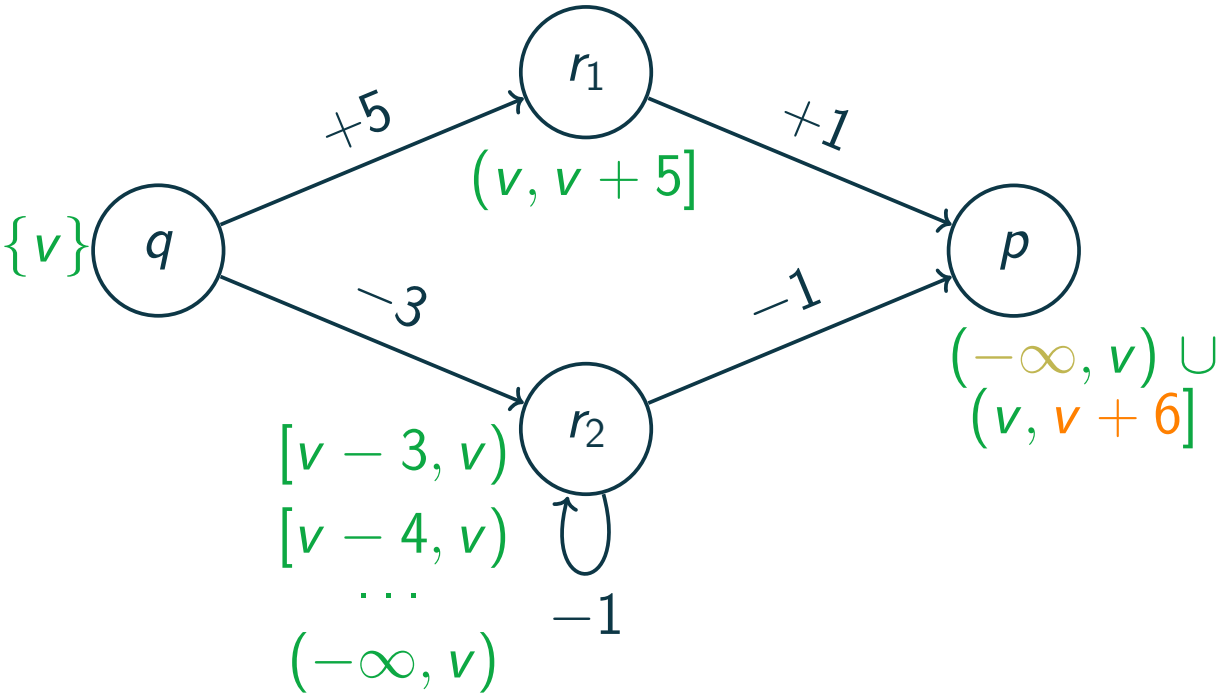

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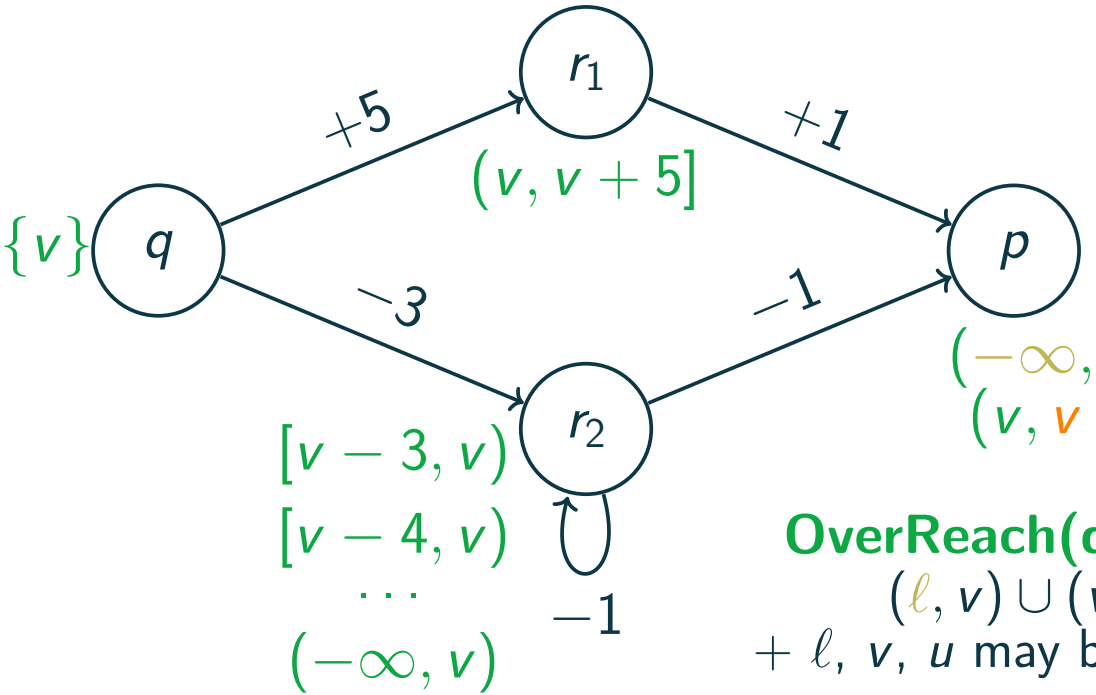

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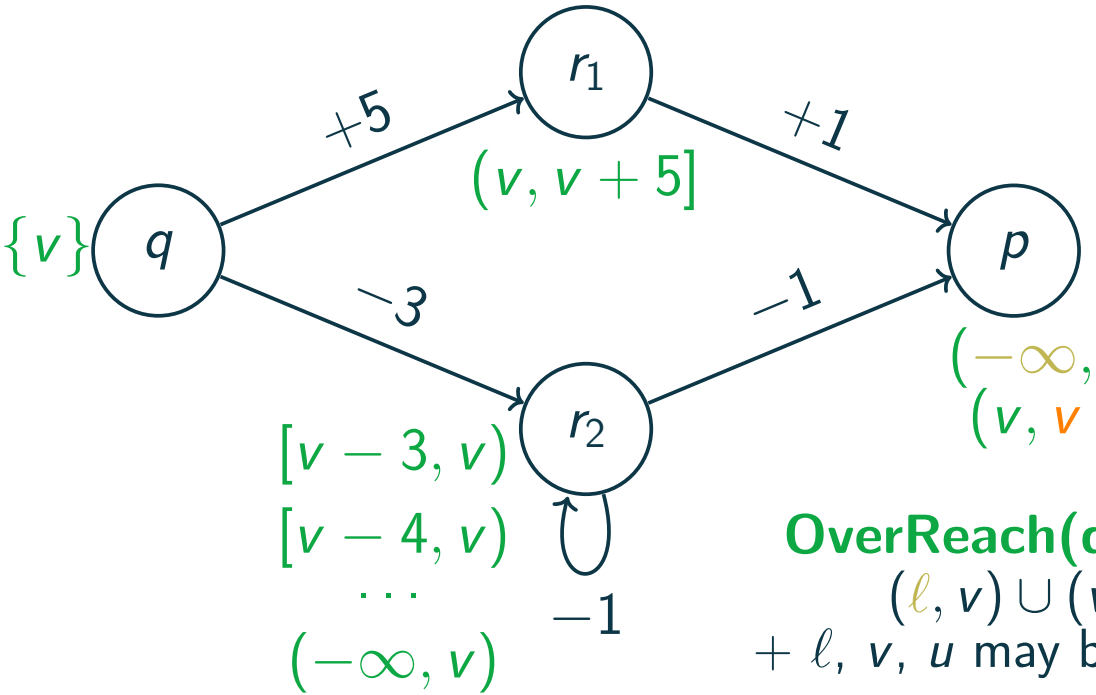

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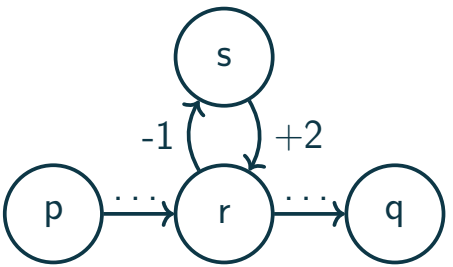

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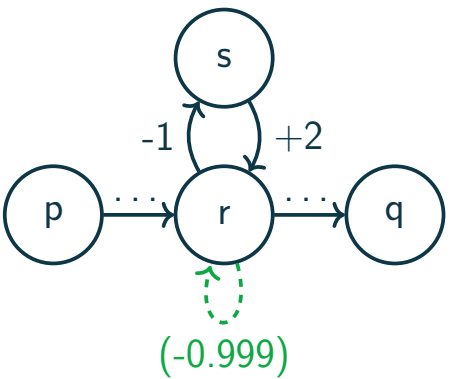

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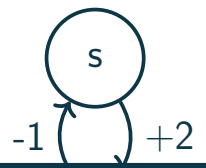

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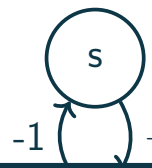

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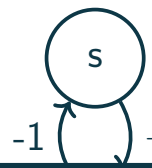

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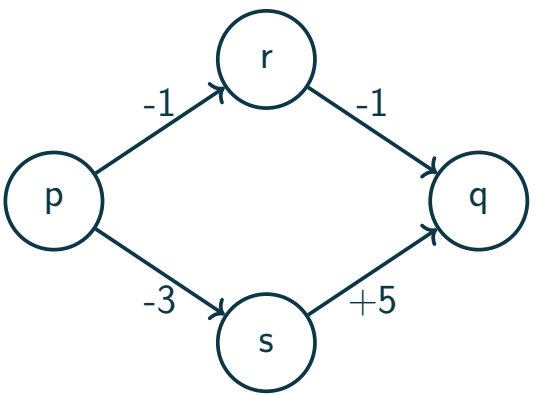

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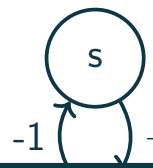

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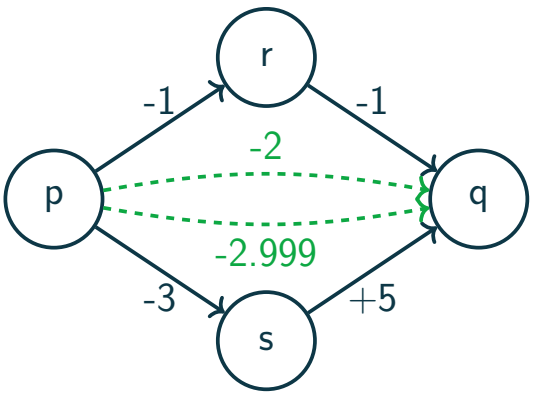

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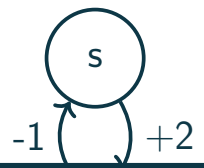

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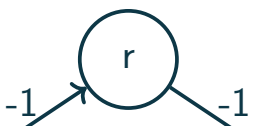

Longest path problem, but:
No cycle with neg. edge
$\Rightarrow \in$ NC^2

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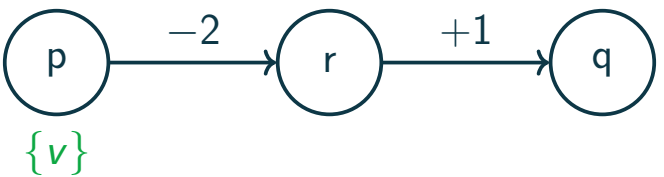

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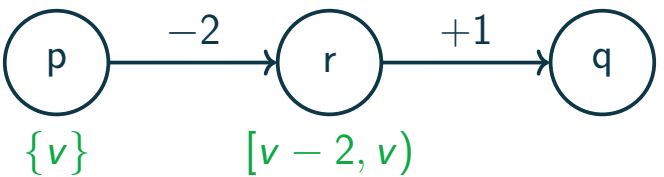

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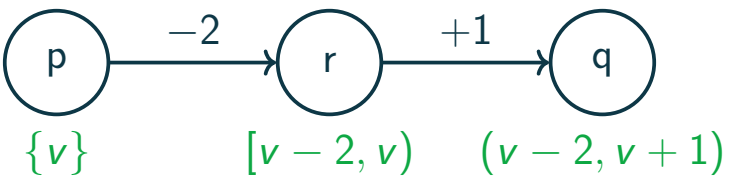

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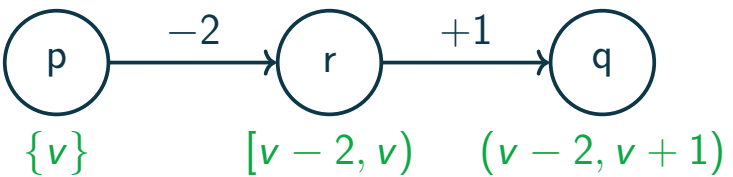

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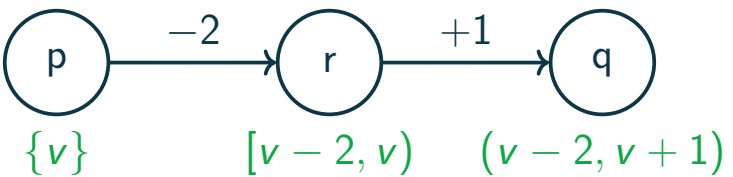

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$\Rightarrow$ Can be checked in $\mathbf{NC^2}$:
Reachability in modified copies of the underlying graph **C**

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$\Rightarrow$ Checking whether a path from q to p
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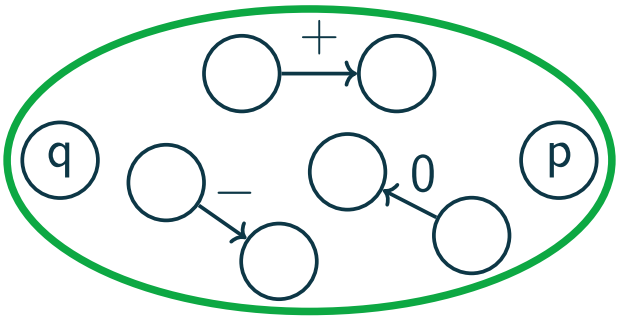

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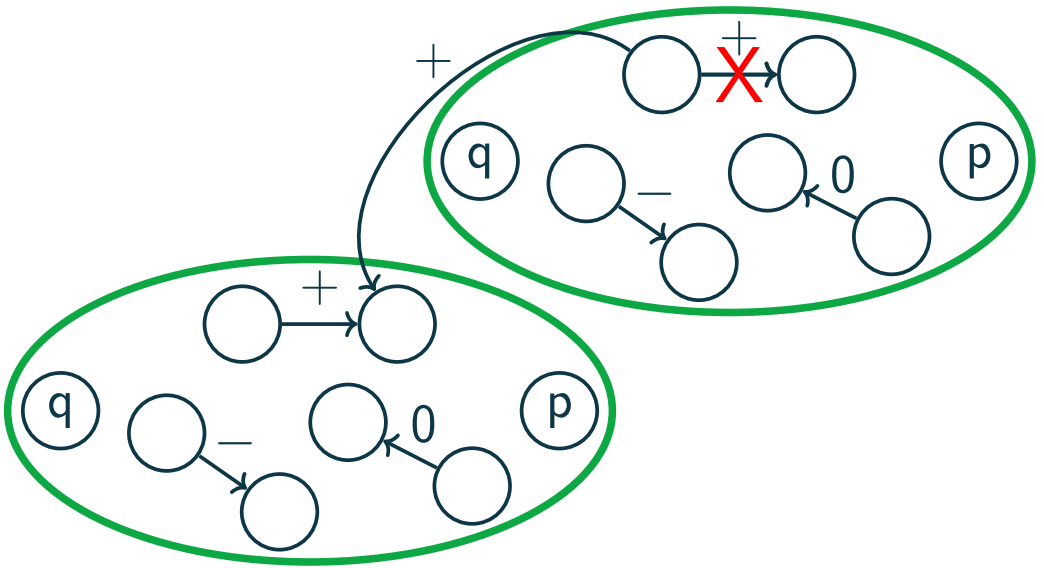

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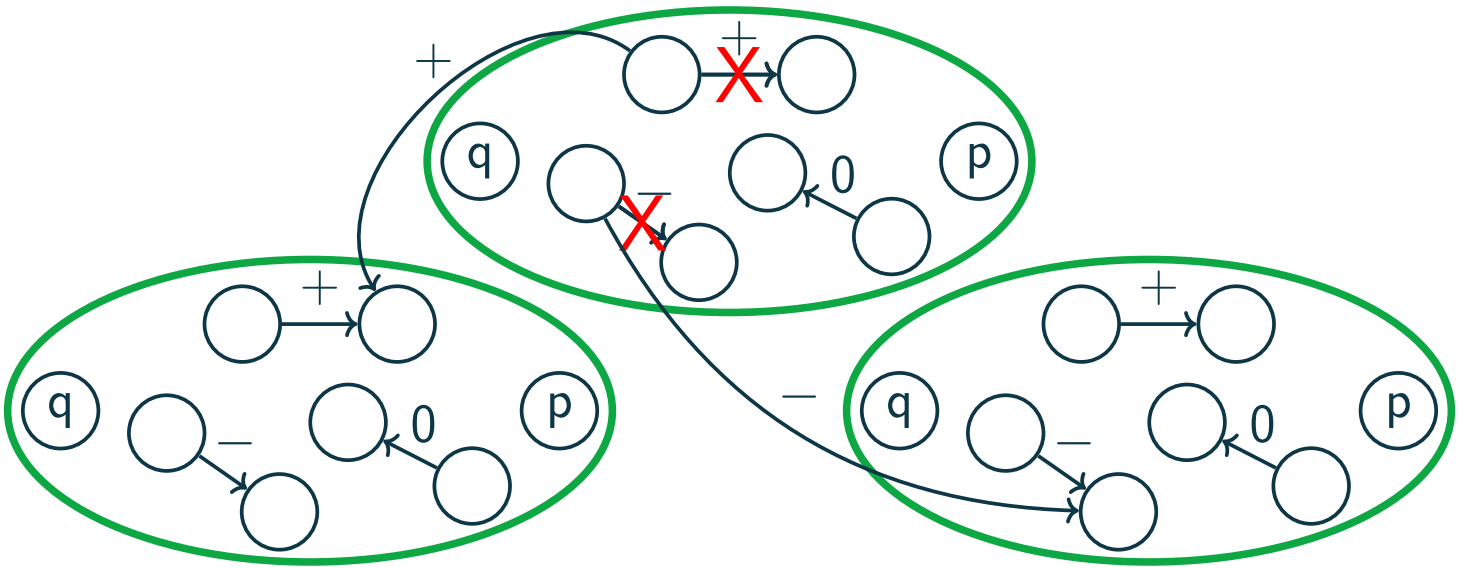

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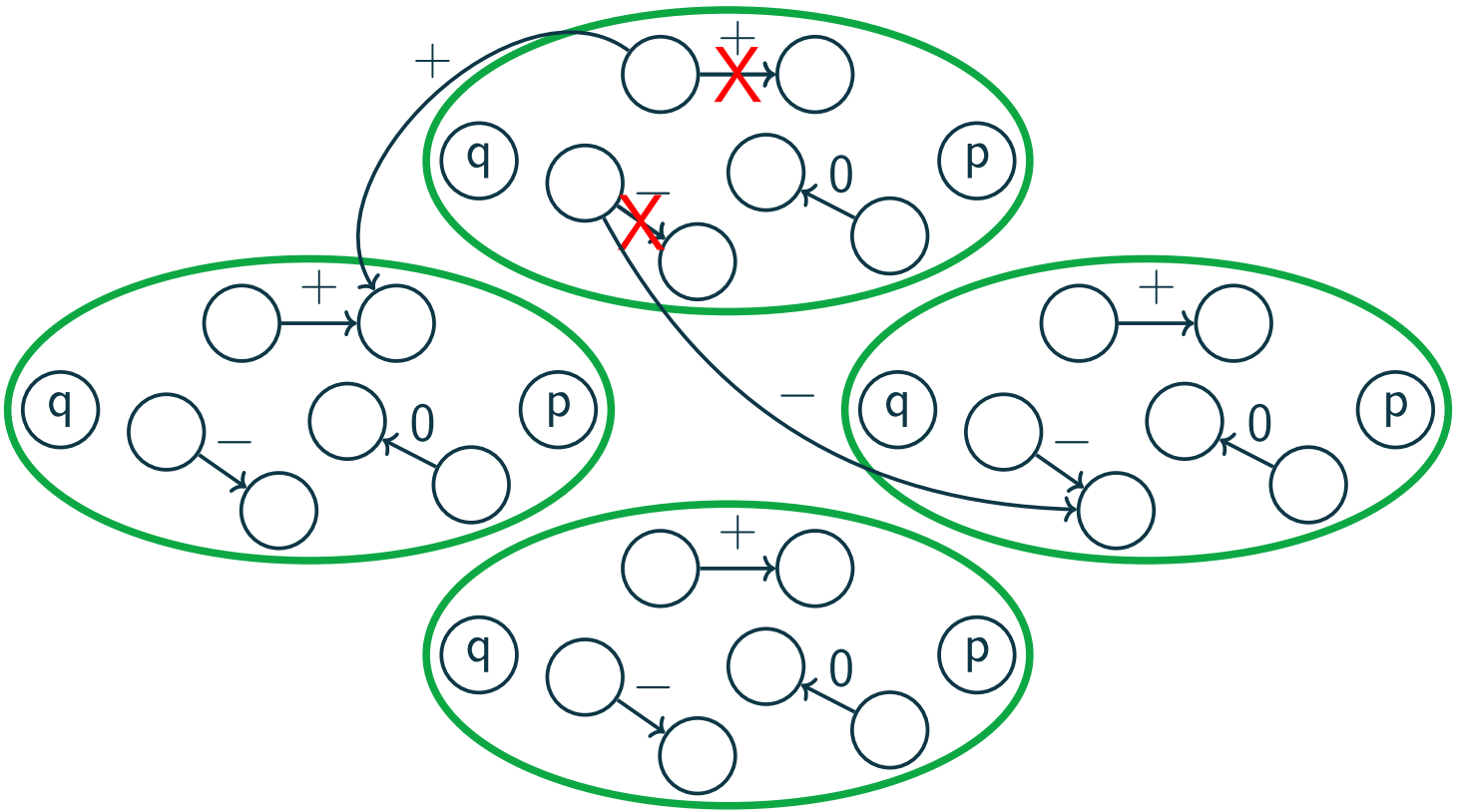

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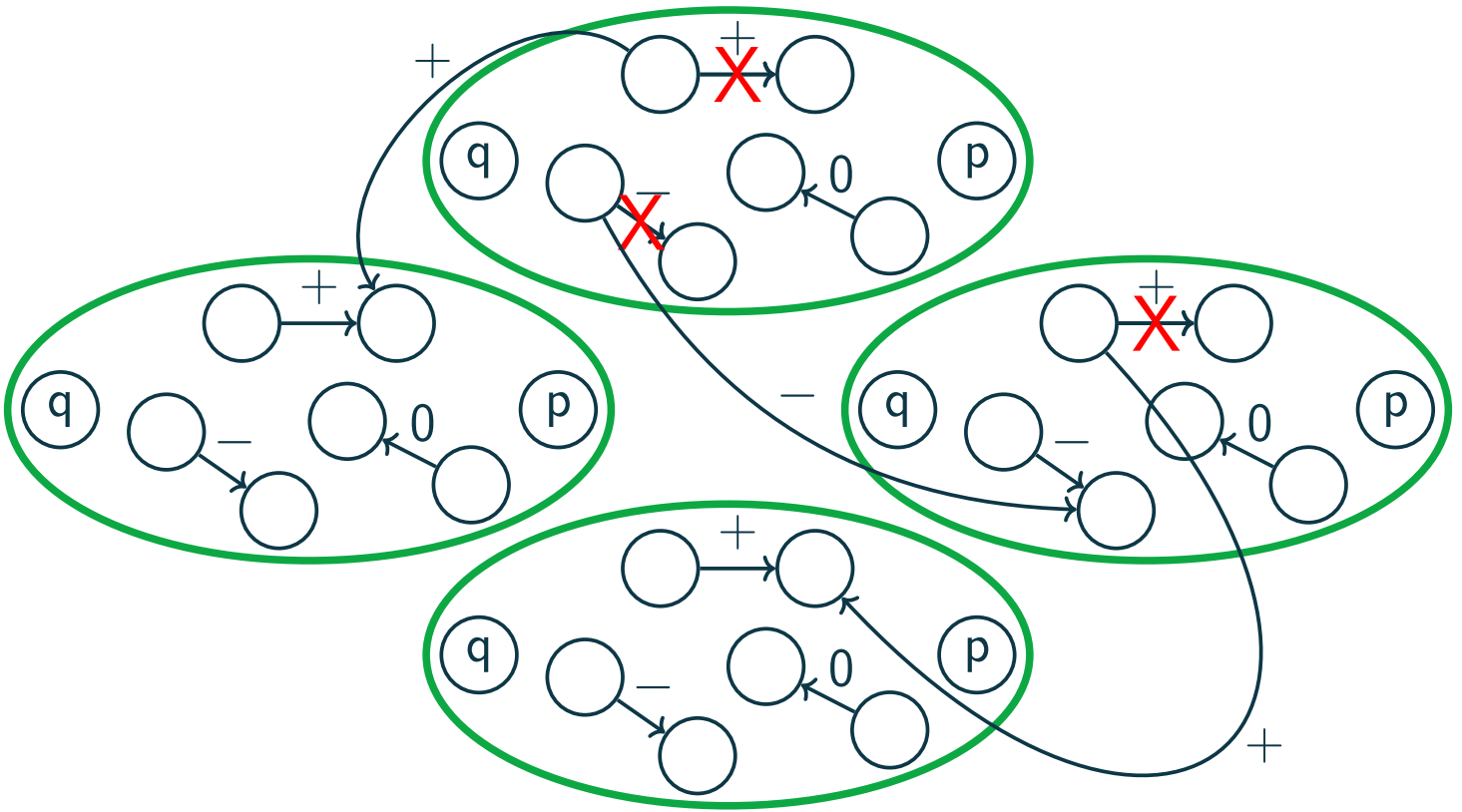

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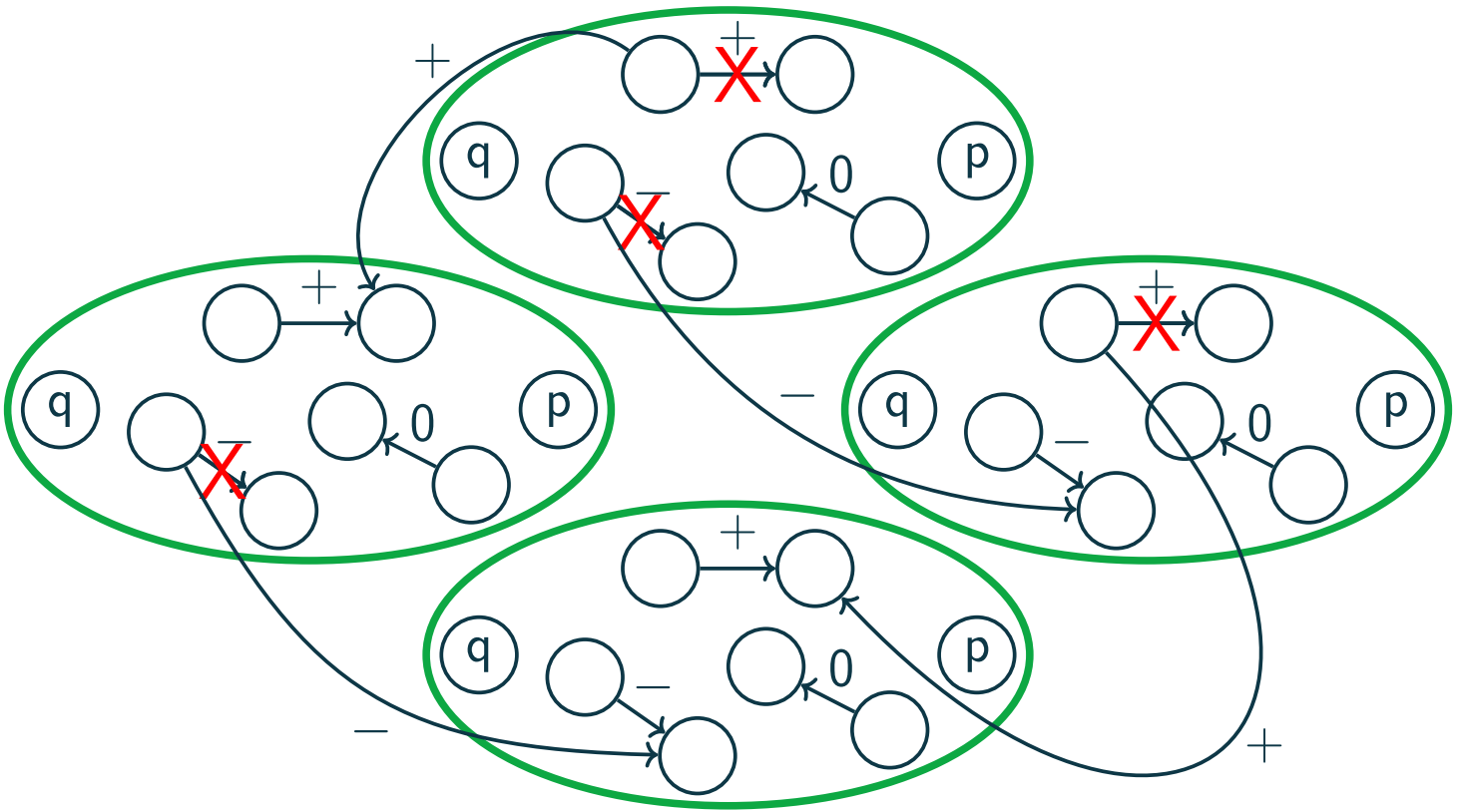

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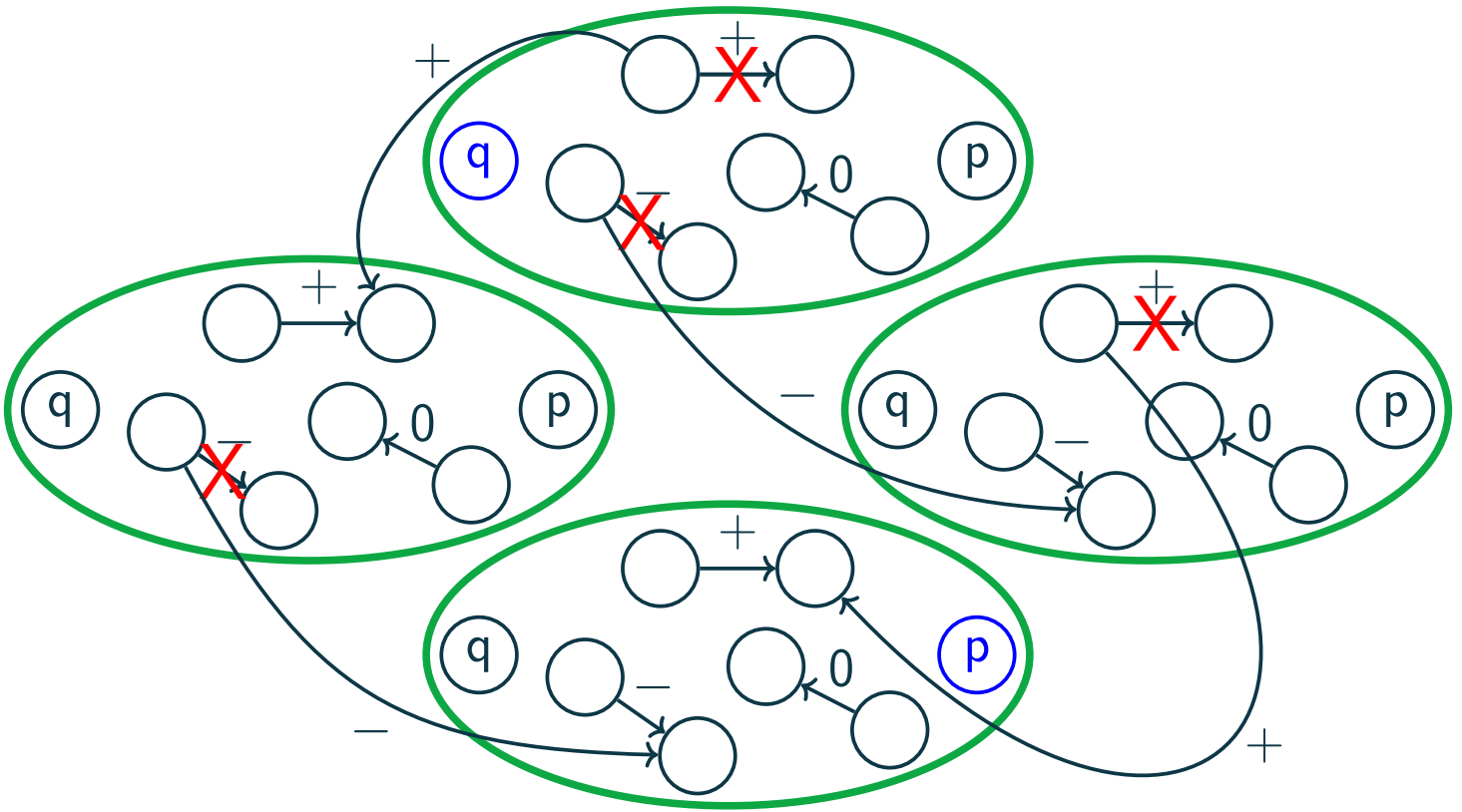

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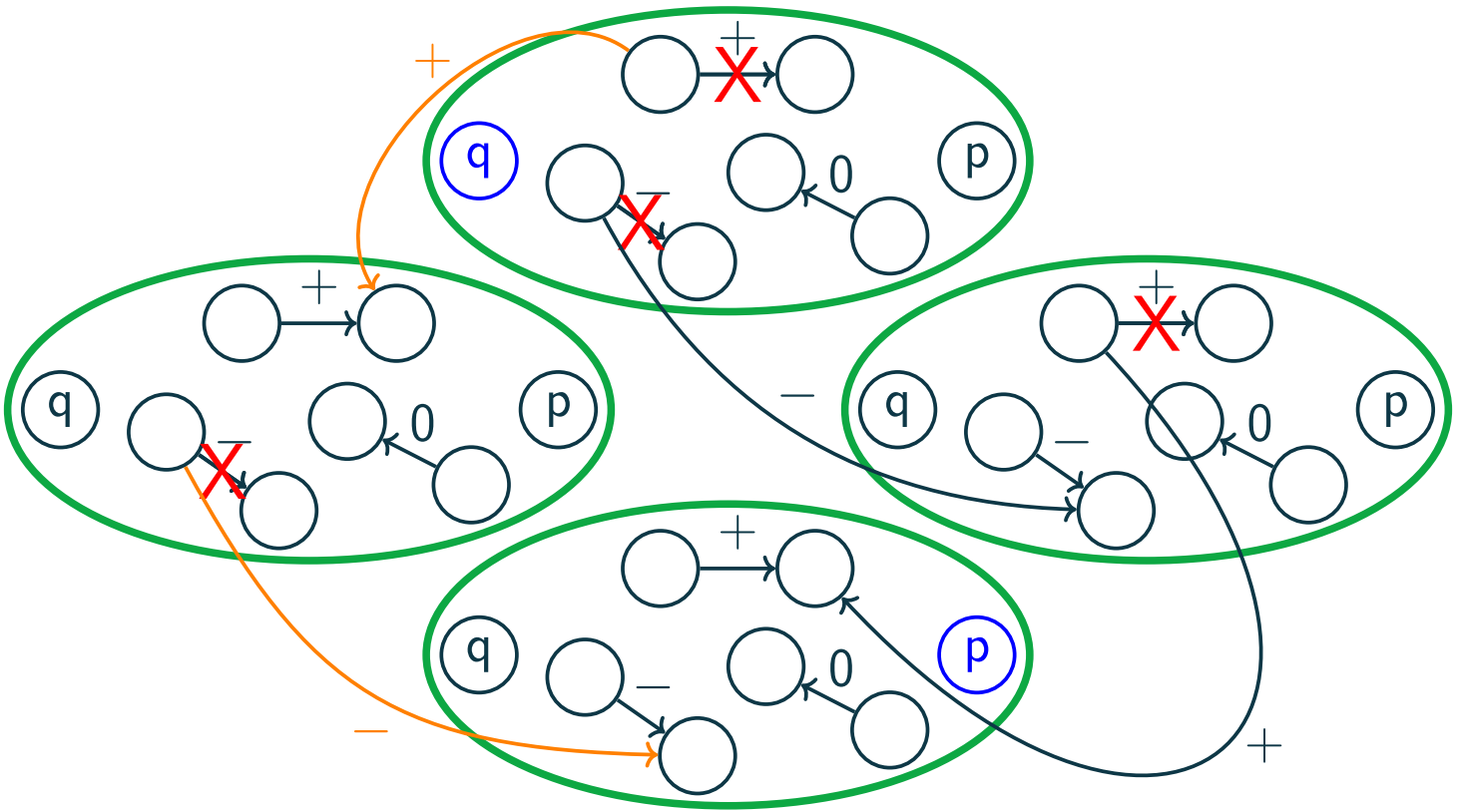

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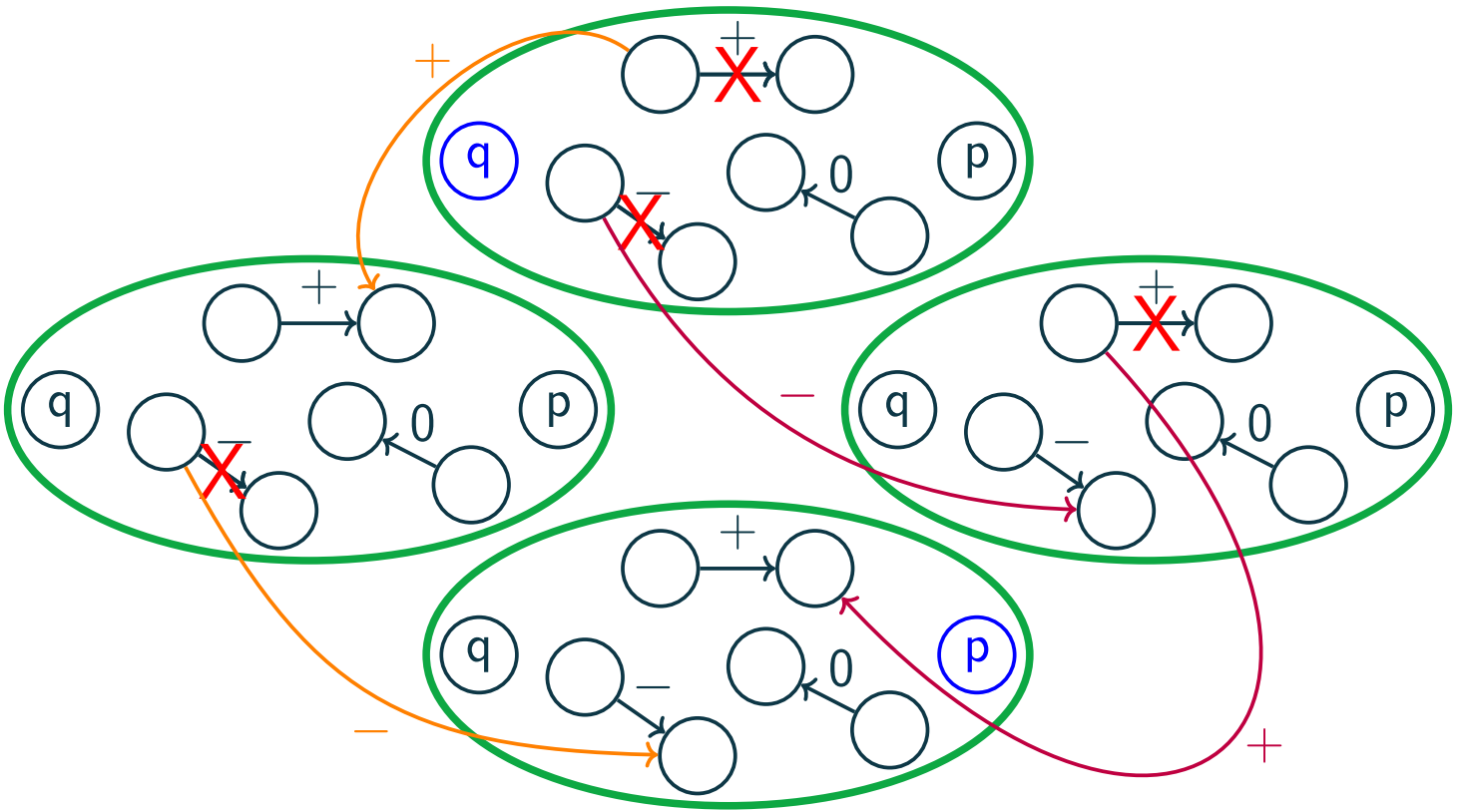

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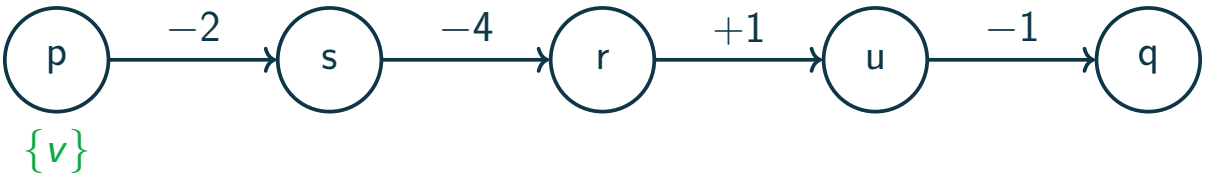

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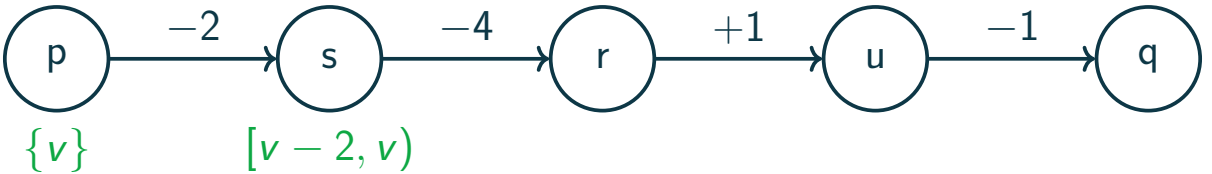

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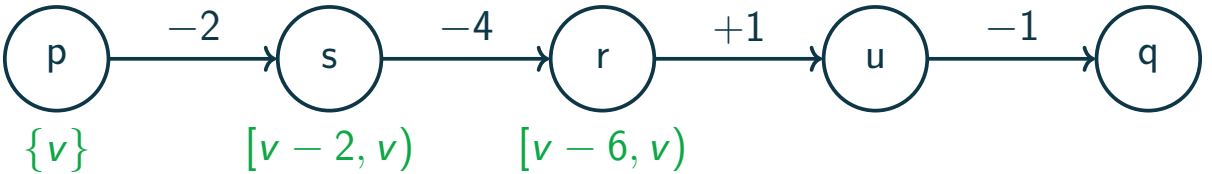

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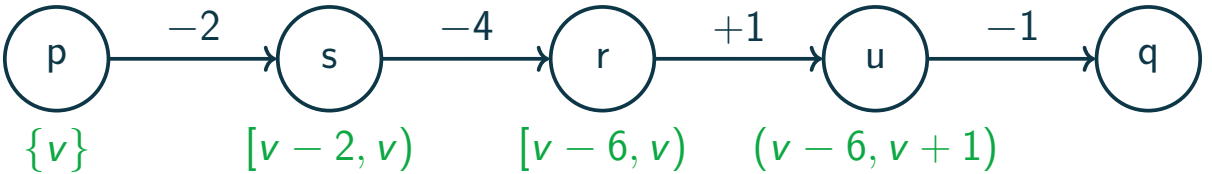

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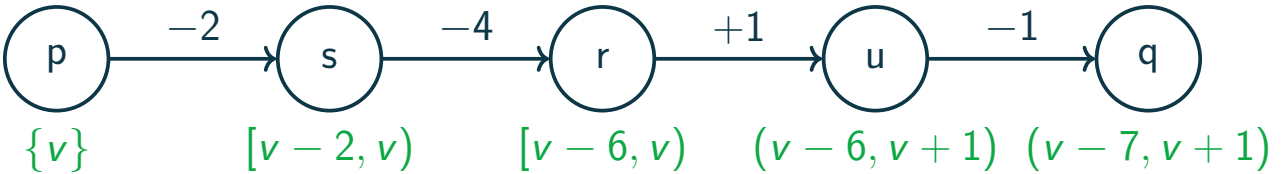

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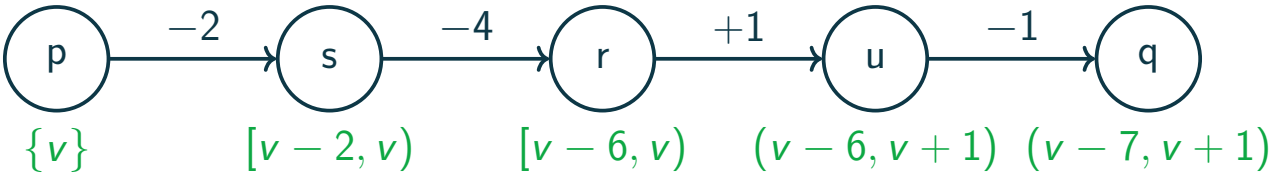

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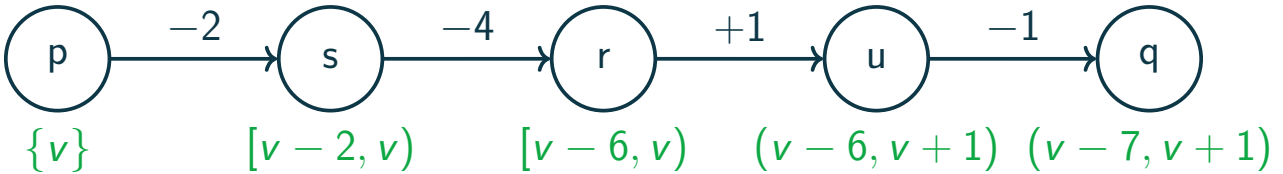

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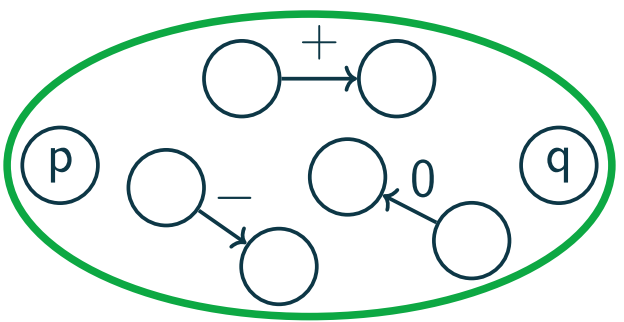

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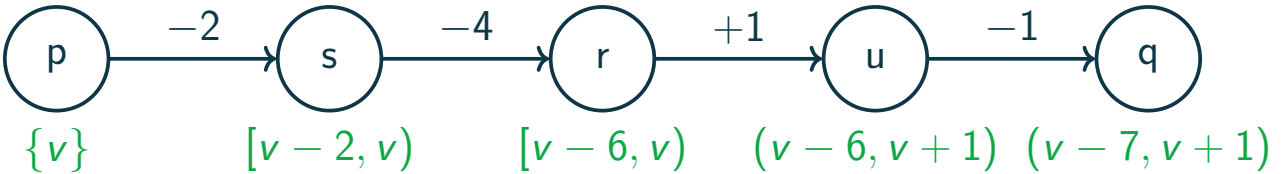

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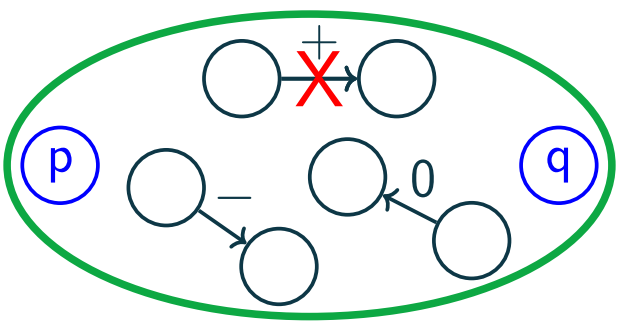
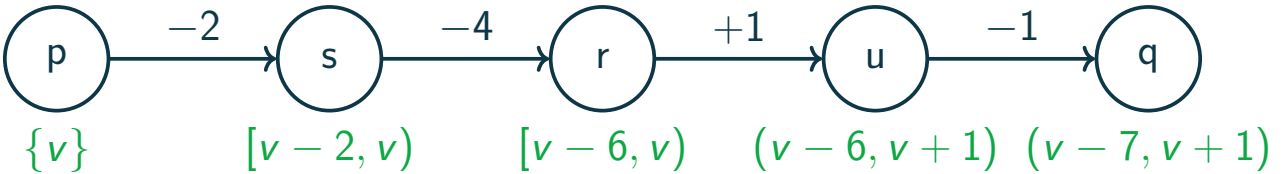
u included if $q(u)$ **reachable** by path with **no neg.** edges



Guardless COCA

1. Computing ℓ and u 2. Checking membership of ℓ, v, u



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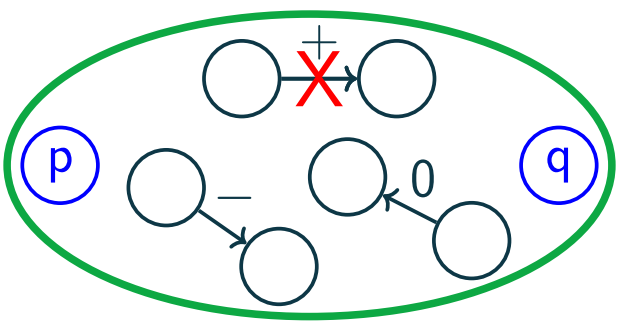

$\Rightarrow$ **Weighted Graph Reachability**!

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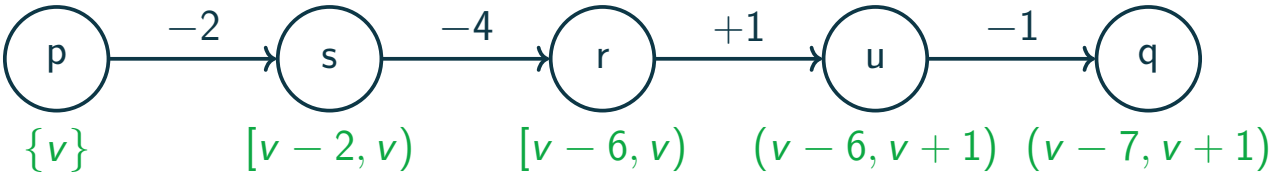

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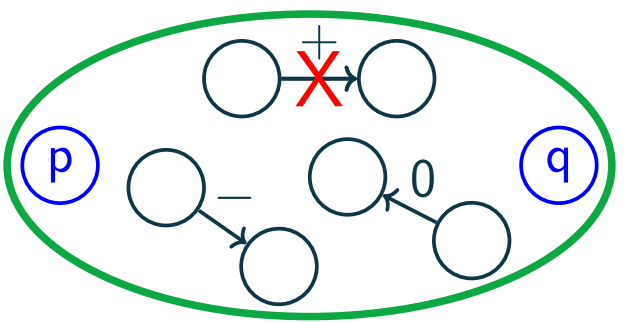

$\Rightarrow$ **Weighted Graph Reachability**!

$\Rightarrow$ **in NC**^{2}!

Reachability in Continuous OCA (COCA)

COCA have much lower complexity

Guardless OCA:
NP-complete
[Haase et al.,'09]

$\Rightarrow$

Guardless COCA:
in **NC^2** (below P-time)
Even with global guards and equality tests

OCA:
PSPACE-complete
[Fearnley,Jurdziński,'15]

$\Rightarrow$

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in **P-time**

Parametric OCA:
Decidability unknown

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More in our paper: Continuous One-Counter Automata, LICS '21

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Overapproximations help when reachability is **intractable**

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Overapproximate **One-Counter Automata** via **Continuous One-Counter Automata**

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Reachability sets of Continuous One-Counter Automata are **unions** of **few** intervals

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This work:

Overapproximate **One-Counter Automata** via **Continuous One-Counter Automata**

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Thanks for listening!